

Bounded Approximation Property for Sobolev spaces on simply-connected planar domain

Maria Roginskaya, Michał Wojciechowski

ABSTRACT

We show that Sobolev space $W_1^1(\Omega)$ of any planar one-connected domain Ω has the Bounded Approximation property. The result holds independently from the properties of the boundary of Ω . The prove is based on a new decomposition of a planar domain.

1. Introduction

In the present paper we study the bounded approximation property (BAP) of Sobolev spaces $W_1^1(\Omega)$ of functions with integrable gradient on a bounded connected simply connected planar domain Ω .

Recall that a Banach space X has the *bounded approximation property* if for every $\varepsilon > 0$ and every finite set $S \subset X$ there exists a finite rank operator $T_\varepsilon : X \rightarrow X$ with norm bounded independent of ε and S , such that $\|T_\varepsilon x - x\| < \varepsilon$ for all $x \in S$.

1.1. Sobolev spaces and BV

Let $\Omega \subset \mathbf{R}^2$ be an open domain and E be a linear (finite dimensional) space. By $M(\Omega, E)$ we denote the space of bounded E -valued measures on Ω with norm $\|\mu\|_M = |\mu|(\Omega)$ where $|\mu|$ denotes the total variation of the measure μ (cf. [DU], Sect. I.1, Def. 4).

By $BV(\Omega)$ we denote the space of all measurable functions summable on Ω , whose distributive first order derivatives are measures from $M(\Omega, \mathbf{R}^2)$. The norm on $BV(\Omega)$ is defined as

$$\|f\|_{BV} = \|f\|_1 + |\nabla f|(\Omega);$$

here the symbol ∇f denotes the distributional gradient of f . By $W_1^1(\Omega)$ we denote the subspace of $BV(\Omega)$ of those functions whose derivatives are absolutely continuous with respect to the 2-dimensional Hausdorff measure on Ω .

Our main result is the following.

THEOREM 1.1. *For every simply connected bounded planar domain Ω the space $W_1^1(\Omega)$ has BAP.*

The BAP property of $W_1^1(\Omega)$ has been already proved for some special classes of the domains (cf. [ACPP]). Typically the requirement on the domain includes some sort of smoothness of its boundary. As far to our knowledge this is a first result of this type which poses no conditions on the boundary of the domain.

The main ingredient of the proof of the above statement is the following geometrical construction.

1.2. Geometric decomposition

DEFINITION 1. We say that a set A whose boundary is a simple curve consisting of intervals and circular arc is η -elementary[†] when for any pair $x, y \in \partial A$ there exists a rectifiable curve $\gamma \subset \partial A$ connecting x with y such that $|\gamma| \leq \eta d(x, y)$.

(We denote by $d(x, y)$ the Euclidean distance between x and y , and by $|\gamma|$ the length of a rectifiable curve γ .)

THEOREM 1.2 (Partition Theorem). *There exists four universal constants $k \in \mathbf{N}$ and $c, C, \eta > 0$, such that following holds for every bounded connected simply connected domain $\Omega \subset \mathbf{R}^2$. For every $\varepsilon > 0$ there exist two finite families \mathcal{A}_ε and \mathcal{B}_ε of open subsets of Ω , such that:*

- (i) A is an η -elementary domain for every $A \in \mathcal{A}_\varepsilon$;
- (ii) For every $A \in \mathcal{A}_\varepsilon$ holds $c\varepsilon \leq \text{diam}(A) \leq C\varepsilon$;
- (iii) every two different sets from $\mathcal{A}_\varepsilon \cup \mathcal{B}_\varepsilon$ are disjoint;
- (iv) for every $A \in \mathcal{A}_\varepsilon$ there are at most k different $A' \in \mathcal{A}_\varepsilon$ such that $\overline{A} \cap \overline{A'} \cap \Omega \neq \emptyset$;
- (v) for every $A \in \mathcal{A}_\varepsilon$ there is at most one $B \in \mathcal{B}_\varepsilon$ such that $\overline{A} \cap \overline{B} \cap \Omega \neq \emptyset$;
- (vi) for every $B \in \mathcal{B}_\varepsilon$ there is exactly one $A = A_B \in \mathcal{A}_\varepsilon$ such that $\overline{B} \cap \overline{A_B} \cap \Omega \neq \emptyset$;
- (vii) $\bigcup_{A \in \mathcal{A}_\varepsilon} \overline{A} \cup \bigcup_{B \in \mathcal{B}_\varepsilon} \overline{B} \supset \Omega$;
- (viii) $(\bigcup_{B \in \mathcal{B}_\varepsilon} B) \cup (\bigcup_{B \in \mathcal{B}_\varepsilon} A_B) \subset (\mathbf{R}^2 \setminus \Omega)_\varepsilon$.

Partition Theorem is of its own interest. It presents decomposition of Ω into "good" pieces \mathcal{A}_ε and "bad" \mathcal{B}_ε which are small and isolated. The cardinalities of \mathcal{A}_ε and \mathcal{B}_ε as functions of ε could be used to quantitative description of how "bad" the shape of a simply connected planar domain is. Connections of this measure of badness with the others, such as for example dimension of the boundary is unknown to us. We postpone the proof of the Partition Theorem to section 3.

1.3. Notation and organization

In our construction we often need to use interior/closure of the same set. To avoid a confusion we denote the closure of a set A as \overline{A} , the interior as A° , and the boundary as ∂A . On a few occasion where it doesn't lead to a confusion we write A for \overline{A} or A° .

For any set K the set K_ε stands for $\{x \in \mathbf{R}^2 : \text{dist}(x, K) < \varepsilon\}$.

If $A \subset \Omega$ is a measurable subset and $\mu \in M(\Omega, E)$, by $\mu \llcorner A$ we denote the measure defined by $(\mu \llcorner A)(B) = \mu(A \cap B)$. By λ_n we denote the n -dimensional Hausdorff measure ($n = 1, 2$).

By γ we denote a simple curve in the plane, and if it is important to us that it has ends a and b we denote it $\gamma(a, b)$. It is a topological fact (Jordan lemma) that if we have two curves $\gamma_1(a, b)$ and $\gamma_2(a, b)$, such that $\gamma_1 \cap \gamma_2 = \{a, b\}$, then those two curves together is the boundary of a bounded connected domain, which we denote $\Omega_{\gamma_1, \gamma_2}$.

The organization of the paper is as follows. In Section 2 we prove Theorem 1.1 assuming Partition Theorem. Proof uses mainly trace estimate due to Gagliardo together with the Poincare inequality for regular domains.

In Section 3 we give the proof of Partition Theorem 1.2.

[†]Notice that domains with this property are Lavrentiev domains, but as we need explicit value of the constant involved, it seems as a less of mouthful to call such domains η -elementary

2. Proof of Theorem 1.1

2.1. Averaging Operator

An implication of the Partition Theorem is the boundedness of the averaging operator defined below.

Let $f \in W_1^1(\Omega)$ and $A \in \mathcal{A}_\varepsilon$. We use the notation $f_A = \lambda_2(A)^{-1} \int_A f d\lambda_2$. By $\text{tr}_A f$ we denote the Galiardo trace of function $f \in W_1^1(A)$ or $f \in BV(A)$ on the boundary ∂A (here and everywhere else in the article we refer to [Z] for standard facts about the Galiardo trace).

Note that since the boundary of an elementary domain A is a union of intervals and circular arcs, the definition of the trace operator is identical to the definition of the trace on a line.

If $\Lambda \subset \partial A$ is a λ_1 -measurable set of positive λ_1 measure we put $g_\Lambda = \lambda_1(\Lambda)^{-1} \int_\Lambda g d\lambda_1$ for every function g integrable on Λ .

For $B \in \mathcal{B}_\varepsilon$ we set $\Lambda(B) = \partial B \cap \partial A_B \cap \Omega$. Note that on $\Lambda(B)$ we have $\text{tr}_B f|_B = \text{tr}_{A_B} f|_{A_B}$ for $f \in W_1^1(\Omega)$, as the left and right traces coincide inside the domain (see [Z]).

For $f \in BV(\Omega)$ we put

$$Tf(x) = \begin{cases} f_A & \text{for } x \in \text{int} A \\ (\text{tr}_B f)_{\Lambda(B)} \cdot h_B(x) & \text{for } x \in \text{int} B \end{cases} \quad (1)$$

where h_B is any fixed for the given decomposition function from $BV(B)$, chosen so that

$$\begin{aligned} \|h_B\|_{BV(B)} + \int_{\Lambda(B)} |\text{tr}_B h_B - 1| d\lambda_1 \\ \leq 2 \inf_{g \in BV(B)} (\|g\|_{BV(B)} + \int_{\Lambda(B)} |\text{tr}_B g - 1| d\lambda_1). \end{aligned}$$

THEOREM 2.1. *For any decomposition of Ω provided by the Partition theorem the operator $T : W_1^1(\Omega) \rightarrow BV(\Omega)$ defined by (1) is a bounded operator with norm depending only on k and η .*

In order to prove Theorem 2.1 we need several lemmas.

LEMMA 2.2. *Let $A \subset \mathbf{R}^2$ is η -elementary domain of diameter 1. Then, there exists $C = C(\eta) > 0$ such that for every $f \in BV(A)$,*

$$\|\text{tr}_A f\|_{L_1(\partial A)} \leq C(\|\nabla f\|_{M(A)} + \|f\|_{L^1(A)}).$$

Proof. This would be a particular case of Theorem 5.10.7 in [Z], except we want the constant depend only from η , but not otherwise from the shape of A .

We show how to derive the above estimate from a similar bound for the unit disk (for which the constant is fixed).

The definition of an η -elementary domain implies that there exists η -bi-Lipschitz function $\psi : \mathbf{T} \rightarrow \partial A \subset \mathbf{R}^2$. By theorems 7.9, 7.10 in [Pom], it has M -bi-Lipschitz extension $\psi : \mathbf{R}^2 \mapsto \mathbf{R}^2$, which in particular maps the unit disc D to A , with bi-Lipschitz constant M depending only from η . Let $T : BV(A) \rightarrow BV(D)$ be defined by $Tf(x) = f \circ \psi(x)$ for $x \in D$. Then, denoting by $J\psi$ a determinant of the Jacobi matrix of ψ , we get

$$\begin{aligned} \|Tf\|_{L^1(D)} &= \int_D |f \circ \psi(x)| d\lambda_2(x) \\ &= \int_A |f(y)| \cdot |J\psi^{-1}(y)| d\lambda_2(y) \leq M^2 \|f\|_{L^1(A)}, \end{aligned}$$

and

$$\|\nabla(Tf)\|_{M(D)} = |\nabla(f \circ \psi)|(D) \leq M \cdot |(\nabla f) \circ \psi|(D) = M \cdot |\nabla f|(A) = M \cdot \|\nabla f\|_{M(A)} \quad (2)$$

Since the composition with a bi-Lipschitz map ψ is an isomorphism from $L^1(\partial A)$ onto $L^1(\mathbf{T})$ with norm depending on η only for the traces, we see that

$$\begin{aligned} \|\mathrm{tr}_A f\|_{L^1(\partial A)} &\leq C(\eta) \|\mathrm{tr}_{\mathbf{T}} Tf\|_{L^1(\mathbf{T})} \leq CC(\eta)(\|\nabla Tf\|_{M(D)} + \|Tf\|_{L^1(D)}) \leq \\ &CM^2C(\eta)(\|\nabla f\|_{M(A)} + \|f\|_{L^1(A)}), \end{aligned}$$

thus proving the lemma. □

LEMMA 2.3. *Let $A \in \mathbf{R}^2$ be an η -elementary domain with diameter 1. Then, there exists $C = C(\eta) > 0$ such that for every $f \in BV(A)$,*

$$\|f - f_A\|_{L^1(A)} \leq C\|\nabla f\|_{M(A)}.$$

Proof. This is essentially the Poincare lemma. The independence of the constant C from anything but η is a well known fact, which is proven for example in [Mar]. We give a proof for the sake of completeness.

Consider the space \dot{BV} , which is a space of all measurable locally summable functions whose distributive first order derivative is a finite measure. The norm on \dot{BV} is defined by $\|f\|_{\dot{BV}} = \|\nabla f\|_M$, with the corresponding factorization over the constants.

Let ψ be the mapping from the previous lemma and T the corresponding operator $Tf = f \circ \psi$, $T : \dot{BV}(A) \mapsto \dot{BV}(D)$. By (2) we see that the norm of the operator T is bounded by a constant which depends only from η .

Denote by $\tilde{T}^{-1} : \dot{BV}(\mathbf{R}^2) \rightarrow \dot{BV}(\mathbf{R}^2)$ given by $\tilde{T}^{-1}g = g \circ \psi^{-1}$. In the similar fashion we see that the norm of \tilde{T}^{-1} is bounded by a constant which depends only from η .

Let further $S : \dot{BV}(D) \rightarrow \dot{BV}(\mathbf{R}^2)$ denote an extension operator defined as

$$S(f) = \begin{cases} f - f_D, & \text{on } D; \\ 0, & \text{on } \mathbf{R}^2 \setminus D. \end{cases}$$

For this operator, $\|Sf\|_{\dot{BV}(\mathbf{R}^2)} \leq \|Sf\|_{BV(\mathbf{R}^2)} \leq C\|f - f_D\|_{BV(D)} \leq C_1\|f\|_{\dot{BV}(D)}$, the second inequality being particular case of Lemma 5.10.4 in [Z], and the last follows to the classical Poincare inequality for the unit disc.

Finally, let $R : \dot{BV}(\mathbf{R}^2) \rightarrow \dot{BV}(2D)$ the operator of restriction to doubled disc, which is obviously has the norm 1.

We consider the extension operator

$$Q := R \circ \tilde{T}^{-1} \circ S \circ T : \dot{BV}(A) \rightarrow \dot{BV}(2D)$$

whose norm depends from η only.

Now we use the Theorem 5.12.7 of [Z].

THEOREM. *Let Ω be a connected admissible domain in \mathbb{R}^n and let μ be a non-trivial positive Radon measure such that $\mathrm{spt} \mu \subset \overline{\Omega}$ and for some constant $M > 0$ that $\mu(B(x, r)) \leq Mr^{n-1}$ for all balls $B(x, r)$ in \mathbb{R}^n . Then there exists a constant $C = C(\Omega)$ such that for each $u \in BV(\Omega)$,*

$$\|u - \bar{u}\|_{L^2(\Omega)} \leq C \frac{M}{\mu(\Omega)} \|\nabla u\|_{M(\Omega)},$$

where $\bar{u} = \frac{1}{\mu(\Omega)} \int u(x) d\mu(x)$.

We apply the Theorem with $\Omega = 2D$ and $\mu = \lambda_2 \lfloor A$ (wlog we may assume $A \subset 2D$).

It is not difficult to see that $\mu(B(x, r)) \leq Cr$ for every $x \in \mathbf{R}^2$, $r > 0$ and some absolute constant $C > 0$. Thus, as $c(\eta) \leq \lambda_2(A) \leq C(\eta)$, the theorem claims that for $g \in BV(2D)$ holds,

$$\|g - g_A\|_{L^1(2D)} \leq \sqrt{C_1} \|g - g_A\|_{L^2(2D)} \leq C_2 \|\nabla g\|_{M(2D)},$$

where the constant C_2 depends only from $\lambda_2(A)$ and, thus, has an upper bound which depends only from η .

It remains to observe that $BV(2D) = BV(2D)$ (though the identification is not bounded) and that for $f \in BV(A)$ holds $(Qf) \lfloor A = f - c_f$. Thus,

$$\begin{aligned} \|f - f_A\|_{L^1(A)} &= \|Qf - (Qf)_A\|_{L^1(A)} \leq \|Qf - (Qf)_A\|_{L^1(2D)} \\ &\leq C_2 \|\nabla(Qf)\|_{M(2D)} = C_2 \|Qf\|_{BV(2D)} \leq C_2 \|Q\| \cdot \|f\|_{BV(A)} = C_3 \|\nabla f\|_{M(A)}. \end{aligned}$$

□

Combining Lemma 2.3 and Lemma 2.2 we get following.

LEMMA 2.4. *Let $A \subset \mathbf{R}^2$ is η -lipschitz domain. Then, there exists $C = C(\eta) > 0$ such that for every $f \in BV(A)$,*

$$\|\text{tr}_A f - f_A\|_{L^1(\partial A)} \leq C \|\nabla f\|_{M(A)}.$$

Proof. As both parts re-scale of order 1 when one changes the size of A , we may assume that diameter of A is 1.

Consider $g = f - f_A$. Using Lemma 2.2 we see that

$$\begin{aligned} \|\text{tr}_A f - f_A\|_{L^1(\partial A)} &= \|\text{tr}_A g\|_{L^1(\partial A)} \\ &\leq C(\|\nabla g\|_{M(A)} + \|g\|_{L^1(A)}) \\ &= C(\|\nabla f\|_{M(A)} + \|f - f_A\|_{L^1(A)}). \end{aligned}$$

Applying now Lemma 2.3 to the second term on the right side we derive the desired conclusion. □

Next three lemmas will be used to prove of the boundedness of the averaging operator.

LEMMA 2.5. *There exists $C = C(\eta) > 0$ with the following property. Let $A, A' \in \mathcal{A}_\varepsilon$ and $\Lambda = \partial A \cap \partial A' \cap \Omega$. Let $D_{A, A'} = (\overline{A \cup A'} \cap \Omega)^\circ$. Then for every $f \in W_1^1(D_{A, A'})$,*

$$\int_\Lambda |f_A - f_{A'}| d\lambda_1 \leq C \|\nabla f\|_{L^1(D_{A, A'})}.$$

Proof. For $f \in W_1^1(D_{A, A'})$ on Λ holds $\text{tr}_\Lambda f = \text{tr}_{A'} f$. Using first the triangle inequality and then Lemma 2.4 we get

$$\begin{aligned} \int_\Lambda |f_A - f_{A'}| d\lambda_1 &\leq \int_\Lambda |\text{tr}_A f - f_A| d\lambda_1 + \int_\Lambda |\text{tr}_{A'} f - f_{A'}| d\lambda_1 \\ &\leq \int_{\partial A} |\text{tr}_A f - f_A| d\lambda_1 + \int_{\partial A'} |\text{tr}_{A'} f - f_{A'}| d\lambda_1 \\ &\leq C \|\nabla f\|_{L^1(A)} + C \|\nabla f\|_{L^1(A')} \\ &\leq C \|\nabla f\|_{L^1(D_{A, A'})} \end{aligned}$$

□

The following two lemmas will be used to estimate the behavior of the averaging operator T near the common boundary of two pieces of which only one is η -Lipschitz.

LEMMA 2.6. *There exists $C = C(\eta) > 0$ such that for every $B \in \mathcal{B}_\varepsilon$ and $f \in W_1^1(\Omega)$,*

$$\int_{\Lambda(B)} |(\operatorname{tr}_B f)_{\Lambda(B)} - f_{A_B}| d\lambda_1 \leq C \|\nabla f\|_{L^1(A_B)},$$

where $\Lambda(B) = \partial B \cap \partial A_B \cap \Omega$.

Proof. Again, as $f \in W_1^1(\Omega)$, we know that $\operatorname{tr}_B f = \operatorname{tr}_{A_B} f$ on $\Lambda(B)$. So,

$$\begin{aligned} \int_{\Lambda(B)} |(\operatorname{tr}_B f)_{\Lambda(B)} - f_{A_B}| d\lambda_1 &= \int_{\Lambda(B)} |(\operatorname{tr}_{A_B} f)_{\Lambda(B)} - f_{A_B}| d\lambda_1 \\ &\quad (\text{notice that the integrand is a constant}) \\ &= \lambda_1(\Lambda(B)) \cdot |(\operatorname{tr}_{A_B} f)_{\Lambda(B)} - f_{A_B}| \\ &= \lambda_1(\Lambda(B)) \cdot |(\operatorname{tr}_{A_B} f - f_{A_B})_{\Lambda(B)}| \\ &= \left| \int_{\Lambda(B)} (\operatorname{tr}_{A_B} f - f_{A_B}) d\lambda_1 \right| \\ &\leq \int_{\Lambda(B)} |\operatorname{tr}_{A_B} f - f_{A_B}| d\lambda_1 \\ &\leq \int_{\partial A_B} |\operatorname{tr}_{A_B} f - f_{A_B}| d\lambda_1 \\ &\quad (\text{which by the Lemma 2.4}) \leq C \|\nabla f\|_{L^1(A_B)}. \end{aligned}$$

□

LEMMA 2.7. *There exists $C = C(\eta) > 0$ such that for every $B \in \mathcal{B}_\varepsilon$ and $f \in W_1^1(\Omega)$,*

$$\begin{aligned} |(\operatorname{tr}_B f)_{\Lambda(B)}| \cdot (\|h_B\|_{BV(B)} + \int_{\Lambda(B)} |\operatorname{tr}_B h_B - 1| d\lambda_1) \\ \leq C (\|f\|_{BV(B)} + \|f\|_{BV(A_B)}). \end{aligned}$$

Proof. Using the definition of h_B , and the fact that, as $f \in W_1^1(\Omega)$, $\operatorname{tr}_B = \operatorname{tr}_{A_B}$ on $\Lambda(B)$, we see that

$$\begin{aligned} &|(\operatorname{tr}_B f)_{\Lambda(B)}| \cdot (\|h_B\|_{BV(B)} + \int_{\Lambda(B)} |\operatorname{tr}_B h_B - 1| d\lambda_1) \\ &\leq 2|(\operatorname{tr}_B f)_{\Lambda(B)}| \inf_{g \in BV(B)} (\|g\|_{BV(B)} + \int_{\Lambda(B)} |\operatorname{tr}_B g - 1| d\lambda_1) \\ &\quad (\text{when } \tilde{g} = |(\operatorname{tr}_B f)_{\Lambda(B)}|g) \\ &= 2 \inf_{\tilde{g} \in BV(B)} (\|\tilde{g}\|_{BV(B)} + \int_{\Lambda(B)} |\operatorname{tr}_B \tilde{g} - (\operatorname{tr}_B f)_{\Lambda(B)}| d\lambda_1) \\ &\leq 2\|f\|_{BV(B)} + 2 \int_{\Lambda(B)} |\operatorname{tr}_B f - (\operatorname{tr}_B f)_{\Lambda(B)}| d\lambda_1 \\ &\quad (\text{and by a triangle inequality}) \\ &\leq 2\|f\|_{BV(B)} + 2 \int_{\Lambda(B)} |f_{A_B} - (\operatorname{tr}_B f)_{\Lambda(B)}| d\lambda_1 + 2 \int_{\Lambda(B)} |\operatorname{tr}_{A_B} f - f_{A_B}| d\lambda_1. \end{aligned}$$

Estimating now the first integral by Lemma 2.6 and the second by Lemma 2.4 we get the desired bound. □

Proof of Theorem 2.1. Let $f \in W_1^1(\Omega)$. Then

$$\begin{aligned} \nabla(Tf) &= \sum_{A, A' \in \mathcal{A}_\varepsilon} \nabla(Tf) \llcorner (\partial A \cap \partial A') \\ &+ \sum_{B \in \mathcal{B}_\varepsilon} \nabla(Tf) \llcorner \Lambda(B) + \sum_{B \in \mathcal{B}_\varepsilon} \nabla(Tf) \llcorner B. \end{aligned} \quad (2.1)$$

If $A, A' \in \mathcal{A}_\varepsilon$ and $\Lambda = \partial A \cap \partial A' \cap \Omega \neq \emptyset$ it is easy to see that ∇Tf is a one-dimensional vector valued measure for which by Lemma 2.5 holds,

$$\|\nabla(Tf) \llcorner (\partial A \cap \partial A')\|_{M(\Omega)} = \int_{\Lambda} |f_A - f_{A'}| d\lambda_1 \leq C \|\nabla f\|_{L^1(A \cup A')}.$$

If $\partial A \cap \partial A' \cap \Omega = \emptyset$ the corresponding term vanishes.

Summing over all neighboring pairs $A, A' \in \mathcal{A}_\varepsilon$, and taking into account that by (iii) every $A \in \mathcal{A}_\varepsilon$ has at most k different neighbors $A' \in \mathcal{A}_\varepsilon$ we get for the first sum in (2.1)

$$\left\| \sum_{A, A' \in \mathcal{A}} \nabla(Tf) \llcorner (\partial A \cap \partial A') \right\|_{M(\Omega)} \leq C \cdot k \cdot \|\nabla f\|_{L^1(\Omega)} \leq C' \|f\|_{W_1^1(\Omega)}.$$

For $B \in \mathcal{B}_\varepsilon$ we get by the triangle inequality

$$\begin{aligned} \|\nabla(Tf) \llcorner \Lambda(B)\|_{M(\Omega)} &= \int_{\Lambda(B)} |(\text{tr}_B f)_{\Lambda(B)} \cdot \text{tr}_B h_B - f_{A_B}| d\lambda_1 \\ &\leq \int_{\Lambda(B)} |(\text{tr}_B f)_{\Lambda(B)} \text{tr}_B h_B - (\text{tr}_B f)_{\Lambda(B)}| d\lambda_1 \\ &\quad + \int_{\Lambda(B)} |(\text{tr}_B f)_{\Lambda(B)} - f_{A_B}| d\lambda_1 \end{aligned}$$

By Lemma 2.6, the second integral on the right hand side is bounded by $C \|\nabla f\|_{L^1(A_B)}$. By Lemma 2.7 the first summand is bounded by $C(\|f\|_{BV(B)} + \|f\|_{BV(A_B)})$. Hence

$$\|\nabla(Tf) \llcorner \Lambda(B)\|_{M(\Omega)} \leq C(\|f\|_{BV(B)} + \|f\|_{BV(A_B)}).$$

Summing over $B \in \mathcal{B}_\varepsilon$ and taking into account that all A_B are distinct we get for the second sum in 2.1

$$\left\| \sum_{B \in \mathcal{B}_\varepsilon} \nabla(Tf) \llcorner \Lambda(B) \right\|_{M(\Omega)} \leq C \cdot \|f\|_{W_1^1(\Omega)}.$$

For the third sum in (2.1) we have

$$\begin{aligned} \left\| \sum_{B \in \mathcal{B}_\varepsilon} \nabla(Tf) \llcorner B \right\|_{M(\Omega)} &= \sum_{B \in \mathcal{B}_\varepsilon} \|\nabla(Tf) \llcorner B\|_{M(\Omega)} \\ &= \sum_{B \in \mathcal{B}_\varepsilon} \|\nabla h_B\|_{M(B)} \cdot |(\text{tr}_B f)_{\Lambda(B)}| \end{aligned}$$

By Lemma 2.7 the last sum is dominated by

$$C \sum_{B \in \mathcal{B}_\varepsilon} (\|f\|_{BV(B)} + \|f\|_{BV(A_B)}) \leq C \|f\|_{W_1^1(\Omega)}.$$

Summing all the three estimates shows $\|\nabla(Tf)\|_{M(\Omega)} \leq C \|f\|_{W_1^1(\Omega)}$. The estimate $\|Tf\|_1 \leq C \|f\|_{W_1^1(\Omega)}$ is trivial, which completes the proof of Theorem 2.1. \square

2.2. Proof of Theorem 1.1

Let $E \subset W_1^1(\Omega)$ be a fixed finite dimensional subspace and $\delta > 0$. Let $\varepsilon > 0$ be such that for every $f \in E$

$$\int_{\Omega \cap (\mathbb{R}^2 \setminus \Omega)_\varepsilon} |f| + |\nabla f| d\lambda_2 < \delta \cdot \|f\|_{W_1^1(\Omega)}$$

Let $\mathcal{A}_\varepsilon, \mathcal{B}_\varepsilon$ be the families of sets provided by Partition Theorem. Let $T_\varepsilon : W_1^1(\Omega) \rightarrow BV(\Omega)$ be the operator given by (1). Let further $\Xi_\varepsilon = \left(\bigcup_{A \in \mathcal{A}_\varepsilon} \overline{A} \cap \Omega \right)^\circ$ and $V_\varepsilon : W_1^1(\Omega) \rightarrow BV(\Omega)$ be

the composition

$$V_\varepsilon = R_\varepsilon \circ (Id - T_\varepsilon),$$

where $R_\varepsilon : BV(\Omega) \rightarrow BV(\Omega)$ is a restriction operator $f \mapsto f\mathbf{1}_{\Xi_\varepsilon}$.

Despite the fact that R_ε itself may not be bounded independently from ε , we can prove that V_ε does. Indeed, the image of $R_\varepsilon \circ (Id - T_\varepsilon)$ is contained in the space

$$Y_\varepsilon = \{f \in BV(\Omega) : f_A = 0 \quad \text{for } A \in \mathcal{A}_\varepsilon, f|_A \in W_1^1(A), \text{ and } f|_B \equiv 0 \text{ for } B \in \mathcal{B}_\varepsilon\}$$

If $f \in Y_\varepsilon$ then

$$\begin{aligned} \|f\|_{BV(\Omega)} &= \|f\|_{L^1(\Omega)} + \sum_{A \in \mathcal{A}_\varepsilon} |\nabla f|(A) + \sum_{A, A' \in \mathcal{A}_\varepsilon} |\nabla f|(\partial A \cap \partial A' \cap \Omega) \\ &\quad + \sum_{B \in \mathcal{B}_\varepsilon} |\nabla f|(\Lambda(B)) \\ &= \|f\|_{L^1(\Omega)} + \sum_{A \in \mathcal{A}_\varepsilon} |\nabla f|(A) + \sum_{A, A' \in \mathcal{A}_\varepsilon} \int_{\partial A \cap \partial A' \cap \Omega} |\text{tr}_A f - \text{tr}_{A'} f| d\lambda_1 \\ &\quad + \sum_{B \in \mathcal{B}_\varepsilon} \int_{\Lambda(B)} |\text{tr}_B f| d\lambda_1 \\ &\leq \|f\|_{L^1(\Omega)} + \sum_{A \in \mathcal{A}_\varepsilon} |\nabla f|(A) + \sum_{A \in \mathcal{A}_\varepsilon} \int_{\partial A} |\text{tr}_A f| d\lambda_1. \end{aligned}$$

The last inequality follows from $\bigcup_{A, A' \in \mathcal{A}_\varepsilon} (\partial A \cap \partial A' \cap \Omega) \cup \bigcup_{B \in \mathcal{B}_\varepsilon} \Lambda(B) = \bigcup_{A \in \mathcal{A}_\varepsilon} \partial A$, with everything but a set of λ_1 -measure zero counted different amount of times on both sides.

Remembering that $f_A = 0$ for $A \in \mathcal{A}_\varepsilon$ and applying Lemma 2.4 we get for $f \in Y_\varepsilon$,

$$\|f\|_{BV(\Omega)} \leq \|f\|_{L^1(\Omega)} + C \sum_{A \in \mathcal{A}_\varepsilon} |\nabla f|(A), \quad (3)$$

where the constant C does not depend from ε , but only from the universal constants k and η .

Now the boundedness of $R_\varepsilon \circ (Id - T_\varepsilon)$ follows from the relations

$$\|R_\varepsilon \circ (Id - T_\varepsilon)f\|_{L^1(\Omega)} \leq \|f\|_{L^1(\Omega)} + \|T_\varepsilon f\|_{L^1(\Omega)}$$

where T_ε is bounded and

$$|\nabla(Id - T_\varepsilon)f|(A) = |\nabla f|(A) \quad \text{for } A \in \mathcal{A}_\varepsilon.$$

Another corollary of formula (3) is the following observation. Let $X_\varepsilon = \left(\bigoplus_{A \in \mathcal{A}_\varepsilon} W_{1,0}^1(A)\right)_{\ell_1}$, where $W_{1,0}^1(A) = \{f \in W_1^1(A) : f_A = 0\}$. The norm of $(g^A)_{A \in \mathcal{A}_\varepsilon} \in X_\varepsilon$ is given by

$$\|(g^A)\|_{X_\varepsilon} = \sum_{A \in \mathcal{A}_\varepsilon} \|g^A\|_{L^1(A)} + \sum_{A \in \mathcal{A}_\varepsilon} |\nabla g^A|(A).$$

By (3) the space Y_ε is isomorphic to X_ε with the norm of the isomorphism dependent only η and not ε , when the isomorphism $J_\varepsilon : Y_\varepsilon \rightarrow X_\varepsilon$ is given by

$$J_\varepsilon : f \mapsto (f|_A)_{A \in \mathcal{A}_\varepsilon} \in X_\varepsilon.$$

Obviously $W_{1,0}^1(A)$, as a complemented subspace of $W_1^1(A)$ of co-dimension one, is isomorphic to the latter with norm non exceeding 2. By [St] Thm 5 Chapt. VI, there exists an extension operator $Ext : W_1^1(A) \rightarrow W_1^1(\mathbb{R}^2)$ with norm depending only on the bi-Lipschitz constant of ∂A . It follows from [PW1] Thm 10 (see also [PW2]) that $W_{1,0}^1(A)$ is isomorphic to $W_1^1(\mathbb{R}^2)$ with uniformly bounded Banach Mazur distance. Therefore X_ε is isomorphic to the l^1 sum of to $W_1^1(\mathbb{R}^2)$ spaces with Banach-Mazur distance independent of ε . Hence, the space X_ε has BAP with constant independent of ε .

By II.E.12b in [Woj] there exists finite rank operator $H_\varepsilon : X_\varepsilon \rightarrow X_\varepsilon$ such that $H_\varepsilon h = h$ for $h \in J_\varepsilon \circ R_\varepsilon \circ (Id - T_\varepsilon)E$ and $\|H_\varepsilon\| \leq C(\eta)$, where $C(\eta)$ depend only from the BAP constant of X_ε and, thus, not from ε .

We have

$$H_\varepsilon \circ J_\varepsilon \circ R_\varepsilon \circ (Id - T_\varepsilon)f = J_\varepsilon \circ R_\varepsilon \circ (Id - T_\varepsilon)f \text{ for } f \in E,$$

It follows that the composition $J_\varepsilon^{-1} \circ H_\varepsilon \circ J_\varepsilon \circ R_\varepsilon \circ (Id - T_\varepsilon) : W_1^1(\Omega) \rightarrow Y_\varepsilon$, which is a finite rank operator, satisfies for $f \in E$,

$$J_\varepsilon^{-1} \circ H_\varepsilon \circ J_\varepsilon \circ R_\varepsilon \circ (Id - T_\varepsilon)f = R_\varepsilon \circ (Id - T_\varepsilon)f. \quad (4)$$

Let us define now $S_\varepsilon : W_1^1(\Omega) \rightarrow BV(\Omega)$ by the formula

$$S_\varepsilon = T_\varepsilon + J_\varepsilon^{-1} \circ H_\varepsilon \circ J_\varepsilon \circ R_\varepsilon \circ (Id - T_\varepsilon)$$

Clearly it is a bounded finite rank operator with norm independent of ε . Moreover, if $f \in E$ then by (4),

$$S_\varepsilon f = T_\varepsilon f + R_\varepsilon \circ (Id - T_\varepsilon)f$$

and therefore

$$\|f - S_\varepsilon f\|_{BV(\Omega)} = \|(Id - R_\varepsilon) \circ (Id - T_\varepsilon)f\|_{BV(\Omega)}. \quad (5)$$

Since

$$(Id - R_\varepsilon) \circ (Id - T_\varepsilon)f = \sum_{B \in \mathcal{B}_\varepsilon} f|_B - (\text{tr}_B f)_{\Lambda(B)} \cdot h_B$$

we get

$$\begin{aligned} \|f - S_\varepsilon f\|_{BV(\Omega)} &= \sum_{B \in \mathcal{B}_\varepsilon} |(\text{tr}_B f)_{\Lambda(B)}| \cdot \int_B |\nabla h_B| + |h_B| d\lambda_2 \\ &\quad + \sum_{B \in \mathcal{B}_\varepsilon} \int_{\Lambda(B)} |\text{tr}_B f - (\text{tr}_B f)_{\Lambda(B)} \cdot \text{tr}_B h_B| d\lambda_1 \\ &\quad + \sum_{B \in \mathcal{B}_\varepsilon} \int_B |\nabla f| + |f| d\lambda_2 \\ &\leq \sum_{B \in \mathcal{B}_\varepsilon} |(\text{tr}_B f)_{\Lambda(B)}| \cdot (\|h_B\|_{BV(B)} + \int_{\Lambda(B)} |1 - \text{tr}_B h_B| d\lambda_1) \\ &\quad + \sum_{B \in \mathcal{B}_\varepsilon} \int_{\Lambda(B)} |\text{tr}_B f - (\text{tr}_B f)_{\Lambda(B)}| d\lambda_1 \\ &\quad + \sum_{B \in \mathcal{B}_\varepsilon} \|f\|_{BV(B)} \end{aligned}$$

Applying now Lemma 2.7 to the first sum we get bound $C \cdot \sum_{B \in \mathcal{B}_\varepsilon} (\|f\|_{BV(B)} + \|f\|_{BV(A_B)})$. For the second sum we get the same bound using the triangle inequality

$$\begin{aligned} \sum_{B \in \mathcal{B}_\varepsilon} \int_{\Lambda(B)} |\text{tr}_B f - (\text{tr}_B f)_{\Lambda(B)}| d\lambda_1 &\leq \sum_{B \in \mathcal{B}_\varepsilon} \int_{\Lambda(B)} |\text{tr}_B f - f_{A_B}| d\lambda_1 \\ &\quad + \sum_{B \in \mathcal{B}_\varepsilon} \int_{\Lambda(B)} |f_{A_B} - (\text{tr}_B f)_{\Lambda(B)}| d\lambda_1 \end{aligned}$$

and Lemma 2.4 and 2.6 respectively.

Therefore we get

$$\begin{aligned} \|f - S_\varepsilon f\|_{BV(\Omega)} &\leq C \cdot \sum_{B \in \mathcal{B}_\varepsilon} (\|f\|_{BV(B)} + \|f\|_{BV(A_B)}) \\ &\leq \int_{\Omega \cap (\mathbb{R}^2 \setminus \Omega)_\varepsilon} |f| + |\nabla f| d\lambda_2 \end{aligned}$$

By our choice of ε , for $f \in E$ we have

$$\|f - S_\varepsilon f\|_{BV(\Omega)} < \delta \cdot \|f\|_{W_1^1(\Omega)}. \quad (6)$$

Operator S_ε is almost what we need to establish BAP with one exception. Instead of $W_1^1(\Omega)$ its rank is contained in $BV(\Omega)$. To remedy that we use the result of [W] which says that $BV(\Omega)$ is contained in the second dual of $W_1^1(\Omega)$ and the variant of the principle of local reflexivity. Namely we have

LEMMA 2.8. (cf. [W], Thm 4) *There is embedding $\delta : BV(\Omega) \rightarrow W_1^1(\Omega)^{**}$ such that $\delta \circ \iota = \chi$ where $\iota : W_1^1(\Omega) \rightarrow BV(\Omega)$ is a natural embedding and $\chi : W_1^1(\Omega) \rightarrow W_1^1(\Omega)^{**}$ is a canonical embedding.*

Put $S'_\varepsilon : W_1^1(\Omega) \rightarrow BV(\Omega)$ by $S'_\varepsilon = S_\varepsilon + \Pi_E \circ \iota - S_\varepsilon \circ \Pi_E \circ \iota$ where $\Pi_E : BV(\Omega) \rightarrow E$ is any projection on E of norm less than $\dim E$. Obviously $S'_\varepsilon f = f$ for $f \in E$ and, by (6),

$$\|S'_\varepsilon f\| = \|S_\varepsilon f + (Id - S_\varepsilon) \circ \Pi_E \circ \iota f\| \leq \|S_\varepsilon f\| + \delta \|\Pi_E \circ \iota f\| \leq (\|S_\varepsilon\| + \delta \dim E) \|f\|,$$

and for each E we can start from choosing $\delta = 1/\dim E$, so that $\|S'_\varepsilon\| \leq C(\eta) + 1$.

Applying now Lemma 2.8, we obtain a finite dimensional space $\delta \circ S'_\varepsilon(W_1^1(\Omega)) \subset W_1^1(\Omega)^{**}$ and the principle of local reflexivity yields the existence a norm 2 operator $U_\varepsilon : \delta \circ S'_\varepsilon(W_1^1(\Omega)) \rightarrow W_1^1(\Omega)$ such that $U_\varepsilon f = f$ for $f \in E$. Hence $U_\varepsilon \circ S'_\varepsilon$ is a bounded finite rank operator whose restriction to E coincide with identity, and a bound on its norm depends only from η and k . This completes the proof of Theorem 1.1.

3. Decomposition of Ω

3.1. More definitions and notation

For two points $a, b \in \mathbf{R}^2$ we will denote the closed interval with ends in a and b by $[a, b]$ (we allow $[a, b]$ to be a point if $a = b$). The open interval with ends in a and b we denote by (a, b) , and the half-open intervals by $[a, b)$ and $(a, b]$. The line passing through the points $a, b \in \mathbf{R}^2$ (where $a \neq b$) will be denoted by \overline{ab} . In both cases whenever the direction matters we suppose it go from a to b .

We denote an open ball with center in x and radius r by $D(x, r)$. The left half-plane of the two parts into which the line \overline{ab} cuts \mathbf{R}^2 will be denoted by $L(\overline{ab})$. We denote the closure of E in the relative topology of Ω as $\overline{E}^\omega = \overline{E} \cap \Omega$.

We will often not distinguish between a parameterized curve and its image on the plane when it does not lead to a confusion. When we consider a union of two (or more) curves we assume it is parameterized in the natural way according to the parameterizations of the curves in the union (the second curve starts where the first ends).

Through this paper we use the notion of polygon slightly more general than usual. Given a finite collection of points $V = (v_k)_1^n$ we say that V are vertices of a *polygon* P , if P is a connected simply connected bounded open planar domain and the closed curve $\gamma_V = \bigcup_{j=1}^n [v_j, v_{j+1}]$ is the boundary of P (here and further we assume $v_{n+1} := v_1$). We would say that γ_V is counterclockwise (or positive) oriented boundary of P if for $j = 1, \dots, n$ for every point x of the interval (v_j, v_{j+1}) there exists $r > 0$ such that $D(x, r) \cup L(\overline{v_j v_{j+1}}) \subset P$. The intervals $[v_j, v_{j+1}]$ will be called *sides* of the polygon P . Notice that we do not exclude the possibility that a side has length 0 or that two or more of consecutive sides belong to the same line. We also do not expect the curve γ_V to be simple, see for example the Figure 1, yet we often mean in the notations a universal covering of $P \cup \partial P$ in which the boundary is simple.

In our construction we will use a special type of polygons (the example on the Figure 1 happens to be of this special type). To introduce the type of polygons we fix a coordinate system, so that we can talk about vertical/horizontal direction.

DEFINITION 2. We denote as $P = P(v_1, \dots, v_{2n})$ a polygon whose vertices $(v_j)_{j=1}^{2n}$ are numbered in the positive direction. We say that $P(v_1, \dots, v_{2n}) \in VH(\alpha, \beta)$ if the following properties are satisfied:

- (i) The sides $[v_{2j+1}, v_{2j+2}]$ are vertical;
- (ii) The sides $[v_{2j}, v_{2j+1}]$ are horizontal;
- (iii) Some sides may have length zero, i.e. $v_k = v_{k+1}$, but always $v_k \neq v_{k+2}$, i.e. there are no two or more adjoint sides of zero length.

- (iv) All sides of $P(v_1, \dots, v_{2n})$ have length bounded by α ;
- (v) For any k the rectangle $Q([v_k, v_{k+1}], P) := P(v_k, v_{k+1}, c, d)$ such that $||[v_{k+1}, c]|| = \beta$ is a subset of P (see Figure 2a). Notice that the vertices are numbered in the positive direction, so the rectangle is uniquely defined even if the boundary curve is not simple.
- (vi) If the angle $\angle(v_{k+1}v_kv_{k-1}) = 180^\circ$ then $[q, v_k] \subset P$ where $\angle(v_{k+1}v_kq) = 90^\circ$ and $d(v_k, q) = \beta$.
- (vii) If the angle $\angle(v_{k+1}v_kv_{k-1})$ (measured in the positive direction, i.e. interior to P) is 270° , then there exists a closed square $Q(v_k, P) := P(v_k, c, d, e)$ with vertical or horizontal sides of length β which consists only of interior points of P , except for v_k (see Figure 2b). Notice that if the angle at v_k is 270° only one quadrant has sides which do not contain the points of ∂P close to v_k , thus the square is uniquely defined.
- (viii) If the angle $\angle(v_{k+1}v_kv_{k-1})$ (measured in the positive direction, i.e. interior to P) is 360° , then there exists two adjoint closed squares $P(v_k, c, d, e)$ with vertical or horizontal sides of length β consisting only of interior points of P , except for v_k (see Figure 2c). Those squares are unique and their union is denote as $Q(v_k, P)$.

3.2. Overview of the construction

The decomposition will be done in the following way.

Our construction depends on a fixed parameter $d = d(\varepsilon) > 0$ which can be chosen to fit ε . We begin with a construction of a polygon $P_d \in VH(3d, \frac{1}{2}d)$, which is "inscribed" in the domain Ω in the following sense.

PROPOSITION 3.1. *Given a domain Ω and a number $d > 0$ there is a polygon $P_d \subset \Omega$, $P_d \in VH(3d, \frac{1}{2}d)$ such that every side of P_d (including zero-length ones) contains a point of $\partial\Omega$.*

Next, we cut the polygon P_d into η -elementary pieces with disjoint interiors of diameter comparable to d to get a family \mathcal{A}_ε , i.e. $\overline{P_d}^\omega = \bigcup_{A \in \mathcal{A}_\varepsilon} \overline{A}^\omega$ (this part of the construction is described in details in section 3.4).

Finally, we divide the reminder $\Omega \setminus \overline{P_d}^\omega$ to get the family \mathcal{B}_ε so that \mathcal{A}_ε and \mathcal{B}_ε fulfil the property ((vii)) of the Partition Theorem. In order to attain the later the collection \mathcal{B}_ε is obtained in the following way: Ω -closure of every component of $\Omega \setminus \overline{P_d}^\omega$ has a non-empty intersection (an arc of ∂P_d) with Ω -closure of only one set from \mathcal{A}_ε (as one will see from the construction of \mathcal{A}_ε). An elements of \mathcal{B}_ε are then the union of the components of $\Omega \setminus \overline{P_d}^\omega$ which has Ω -closure with non-empty intersection with the Ω -closure of the same element of \mathcal{A}_ε - see the Figure 3. The properties ((v)), ((vi)), and the corresponding part of ((iii)) follow immediately from the way of constructing \mathcal{B}_ε .

Observe that $\bigcup_{B \in \mathcal{B}_\varepsilon} B = \Omega \setminus \overline{P_d}^\omega$. We will show that the construction implies that $\bigcup_{B \in \mathcal{B}_\varepsilon} B \subset (\mathbf{R}^2 \setminus \Omega)_\varepsilon$. This gives us the first part of the property ((viii)) for the partition.

The partition \mathcal{A}_ε of the polygon P_d will be based on the following proposition and scaling.

PROPOSITION 3.2. *There exist universal constants $\Delta, \delta, \eta > 0$, such that, given a polygon $P \in VH(3, \frac{1}{2})$, and a collection[†] of points $\mathcal{P} \subset \partial P$, such that for every side of P the set $[v_k, v_{k+1}] \cap P$ is non-empty, we can decompose the polygon as $\overline{P}^\omega = \bigcup \overline{A_m}^\omega$. Here $\mathcal{A} = \{A_m\}$ is*

[†]Those points in our construction are $\partial P \cap \partial\Omega$

a collection of disjoint η -elementary sets of diameter less than Δ and greater than δ . For every A_m the set $\partial A_m \cap \partial P$ is either an arc of ∂P connecting two points of \mathcal{P} or empty.

Now, the properties ((i)), ((ii)) and the remaining part of the property ((iii)) of the Partition Theorem follow immediately. The property ((iv)) follows to the fact that all the sets in \mathcal{A} are η -elementary and have comparable sizes. We explain this implication in the following lemma.

LEMMA 3.3. *For any $\Delta > \delta > 0$ and $\eta > 0$ there exists $k = k(\eta, \delta, \Delta)$, such that if $\{A_n\}$ is an arbitrary collection of disjoint open η -elementary sets for which $\delta < \text{diam}(A_n) < \Delta$, then the closure of any set in $\{A_n\}$ intersects the closures of no more than k sets from the collection.*

Proof. There exists a constant $c = c(\eta)$ such that area of an η -elementary set A is at least $c \cdot (\text{diam}(A))^2$. Consider a point $p \in A_n$. Then every set A_m , such that $\overline{A_n} \cap \overline{A_m} \neq \emptyset$, is contained in a disc with centre in p and radius 2Δ . At the same time the sets $\{A_n\}$ are disjoint and each has the area at least $c\delta^2$. This means that the number of sets neighboring A_n does not exceed $k = \frac{4\pi\Delta^2}{c\delta^2}$. \square

It remains to observe that, as diameters of the sets of \mathcal{A}_ε are at most Δd and $\partial A_B \cap \partial P_d \neq \emptyset$ for $B \in \mathcal{B}_\varepsilon$, we have $\bigcup_{B \in \mathcal{B}_\varepsilon} A_B \subset (\partial P_d + D(0, \Delta d)) \subset (\mathcal{P} + D(0, (\Delta + 3)d)) \subset (\partial\Omega + D(0, (\Delta + 3)d)) \subset (\mathbf{R}^2 \setminus \Omega)_\varepsilon$ when $d < \varepsilon/(\Delta + 3)$, which gives the remaining part of the property (viii).

Thus, Theorem 1.2 will be established as long as we prove Propositions 3.1 and 3.2. Their proofs constitute the rest of the paper.

3.3. Proof of the proposition 3.1

We begin by constructing a (finite) sequence of increasing by inclusion $VH(d, \frac{1}{2}d)$ -polygons $(P_k^d)_{k=1}^N$, $P_k^d \subset \Omega$. Then we will build a $VH(3d, \frac{1}{2}d)$ -polygon P_d with the required properties such that $P_N^d \subset P_d \subset \Omega$.

Fix $x \in \Omega$. Consider the lattice of squares (in a fixed coordinate system) with the side-length d . We denote by Ω_d the union of all closed lattice squares contained in Ω . Assuming that the given d is small enough (less than $\frac{1}{\sqrt{2}}\text{dist}(x, \partial\Omega)$), we have $x \in (\Omega_d)^\circ$. The polygon $P_0^d = \Omega_d(x)$ is the connected component of $(\Omega_d)^\circ$ which contains the point x . It is easy to see that $P_d \supset \Omega_d(x)$ and $\bigcup \Omega_{2^{-k}}(x) = \Omega$. Thus, $\bigcup P_{2^{-k}}$ is an open covering of Ω with $P_{2^{-k}} \subset P_{2^{-(k+1)}}$, so that for k large enough $\Omega \setminus P_{2^{-k}} \subset (\mathbf{R}^2 \setminus \Omega)_\varepsilon$, which gives us the first part of the property (viii).

The rest of the construction can be re-scaled, so we may assume from now on that $d = 1$ and omit this index in our notations.

Directly from the definition it follows that P_0 is a connected polygon with the sides parallel to the axis, whose vertices have integer coordinates. It is a straightforward observation that P_0 has to be simply connected and its boundary is a simple curve (see Figure 4).

Adding if necessary some zero length sides at the integer-coordinates points of ∂P_0 , we can assume that $P_0 \in VH(1, 1)$ and the length of each side is 1 or zero.

The idea of the construction of the sequence (P_n) is that we increase the polygon by "moving its sides out" as long as possible (see Figure 4). This will be done in a sequence of 3 step-procedures, which slightly differs when we build P_1 from P_0 , and when we build further P_{k+1} from P_k , $k \geq 1$. To make the first step we need the following.

LEMMA 3.4. *If $P = P(v_1, \dots, v_{2n}) \in VH(\alpha, \beta)$ and $P \subset \Omega$ then for any j for which $v_j \neq v_{j+1}$, there exists unique pair of points $p, q \in \mathbf{R}^2$ such that $R = P(v_{j+1}, v_j, p, q)$ is an open rectangle, $[p, q]$ contains a point of $\partial\Omega$, and $\overline{R} \setminus [p, q] \subset \Omega$.*

Proof. As $P \subset \Omega$, $[v_j, v_j + 1] \subset \overline{P} \subset \Omega \cup \partial\Omega$. If $[v_j, v_{j+1}]$ contains a point of $\partial\Omega$, we can take $p = v_j, q = v_{j+1}$. Therefore we may assume that $[v_j, v_{j+1}] \subset \Omega$.

Consider the union $R = \cup P(v_{j+1}, v_j, \xi, \zeta)$ of all open rectangles with a side $[v_{j+1}, v_j]$ (in that direction), such that $\overline{P(v_{j+1}, v_j, \xi, \zeta)} \subset \Omega$. Since we keep fixed the orientation all the rectangles are on the "outer" side of the polygon P (but they may intersect P if the length of the side $[v_j, \xi]$ is big enough).

As Ω is bounded, so is R , as $R \subset \Omega$. As an increasing union of open rectangles with a common side, the union R is an open rectangle, and we may say $R = P(v_{j+1}, v_j, p, q)$ (if $R = \emptyset$ then $p = v_j, q = v_{j+1}$).

On one hand, $\overline{R} \setminus [p, q] \subset \cup \overline{P(v_{j+1}, v_j, \xi, \zeta)} \subset \Omega$ where the union is taken over the same collection of rectangles which defines R .

On the other hand, if $[p, q] \cap \partial\Omega = \emptyset$, then the distance from $[p, q]$ to $\partial\Omega$ is strictly positive, and thus there exists ξ', ζ' , such that $P(v_{j+1}, v_j, \xi', \zeta') \supsetneq R$ and $\overline{P(v_{j+1}, v_j, \xi', \zeta')} \subset \Omega$, which contradicts the definition of R . \square

Notice that the rectangle given by the Lemma 3.4 is unique.

Notation. The side $[v_j, v_{j+1}]$ of the rectangle R provided by the Lemma 3.4 will be called its *base* and the opposite side $[p, q]$ - its *top*.

3.3.1. *Construction of P_1 starting from P_0 .* Clearly all sides of P_0 are contained in Ω .

We first construct \tilde{P}_0 by adding to P_0 all rectangles provided by Lemma 3.4 which are build on its non-zero horizontal sides together with the corresponding bases without(!) the bases' endpoints. This means that we increase amount of the vertices of P_0 (see Figure 5).

We claim that \tilde{P}_0 is a polygon, i.e. connected and simply connected, and that $\tilde{P}_0 \in VH(1, 1)$.

To show this we observe that none of the added rectangles intersect neither P_0 nor any other added rectangle. Indeed, if the height of any added rectangle R would exceed 1, then R contains a lattice square Q with the same base and such that $\overline{Q} \subset \Omega$. But then $Q \subset \Omega_d(x)$, and its base can not be a side of P_0 . But R can not intersect P_0 (which consists of the lattice squares) unless its height is greater than 1.

On the other hand, a pair of added rectangles R_1 and R_2 can intersect each other only if their bases are the opposite (horizontal) sides of a lattice square, and each of them contains the top of the other (see Figure 6). But the top of the rectangle R_1 contains points of $\partial\Omega$, and thus can not be part of $\overline{R_2}$, unless it coincides with its top. But, if the closures of the rectangles intersect only at the tops, the rectangles themselves, being open, do not intersect each other.

When we add a rectangle and the side to which it is attached to the polygon, the union is obviously connected. As the union is homotopic to the initial polygon, it remains simply connected. As the sides of \tilde{P}_0 are either the sides of P_0 or the sides of the added rectangles, non of the sides of \tilde{P}_0 is longer than 1. It is also plain to see that adding a rectangle of the width 1 to a horizontal side of $VH(., 1)$ -polygon gives us again a $VH(., 1)$ -polygon. Thus, the resulting polygon \tilde{P}_0 is a $VH(1, 1)$ -polygon with additional property that each of its non-zero horizontal side contains a point from $\partial\Omega$.

In the next step we consider the zero-length horizontal sides of \tilde{P}_0 . They can be of two types: those with corresponding interior angle of the polygon \tilde{P}_0 equal 180° (we let them be), and those with corresponding interior angle of \tilde{P}_0 equal 360° . We deal with the later case at this (second) step of our construction.

If the angle at a zero length horizontal side $[v_j, v_{j+1}]$ is 360° , this means that one of the adjoint sides $[v_{j-1}, v_j]$ and $[v_{j+1}, v_{j+2}]$ is contained in the other one. W.l.o.g. we assume $[v_{j-1}, v_j] \subset [v_{j+1}, v_{j+2}]$.

There are two possibilities:

- (1) $\partial\Omega \cap (v_{j-1}, v_j] = \emptyset$; in this case we remove $v_j = v_{j+1}$ from the sequence of the vertices of the polygon, thus adding to the polygon the half-open interval $(v_{j-1}, v_j]$;
- (2) $\partial\Omega \cap (v_{j-1}, v_j] \neq \emptyset$; in this case we choose $p \in \partial\Omega \cap (v_{j-1}, v_j]$ which is the most close to v_j and replace both vertices $v_j = v_{j+1}$ by p , thus adding to the polygon the half-open interval $(p, v_j]$.

It is easy to see that non of the two operations changes the connectedness or homotopy class of the polygon. Furthermore, if a polygon is a $VH(1, 1)$ -polygon it clearly remains a $VH(1, 1)$ polygon after any of this two operation.

We perform the operation described above on every zero-length horizontal side of \tilde{P}_0 with the interior angle 360° which doesn't contain a point of $\partial\Omega$. In the end of this step we obtain a polygon which we call P_1^* .

Our aim is to construct a sequence of polygons $(P_k)_{k=1}^N$, such that $P_k \in VH(1, \frac{1}{2})$, and for every odd k each horizontal side should contain a point of $\partial\Omega$, except for those of zero-length at which the interior angle of P_k is 180° (the later may or may not contain points of $\partial\Omega$). Also every vertical side of length less than $\frac{1}{2}$ should be adjoint to an angle of at least 270° (we call this additional conditions *-condition). For an even k the same additional conditions should be satisfied with horizontal replaces by vertical and vice versa. It is the second part of the *-condition that P_1^* may not satisfy.

To obtain the polygon P_1 it remains to take care of the short vertical sides. As $P_1^* \in VH(1, 1) \subset VH(1, \frac{1}{2})$, if a side $[v_j, v_{j+1}]$ is shorter than $\frac{1}{2}$ the angles at v_j and v_{j+1} can not both be 90° . If the one of them is 270° or greater, then the side poses no problem. Consider the last remaining case when one of the angles is 180° . W.l.o.g. we may assume that it is the angle at v_j which is 180° . This means that the side is adjoint to a horizontal side of zero-length, i.e. $v_j = v_{j-1}$. In this case we remove both vertices v_j and v_{j-1} from the sequence of the vertices so that the side $[v_j, v_{j+1}]$ merges with the previous vertical side. Repeating this operation as many times as needed, we get a $VH(\cdot, \frac{1}{2})$ - polygon which satisfies the rest of the conditions, but may have some vertical sides of length greater than 1. We remedy this last problem by adding some zero-length horizontal sides to subdivide the longer sides in pieces of length greater than $\frac{1}{2}$ but not greater than 1, and thus obtain the polygon P_1 .

3.3.2. Construction of P_{k+1} Assume now that we have constructed $P_k \in VH(1, \frac{1}{2})$, which satisfies *-condition. W.l.o.g. we may assume that k is odd, so that every horizontal side of P_k (except for some zero-length with the adjoint angle of 180°) contains a point of $\partial\Omega$, and every vertical side of length less than $\frac{1}{2}$ is adjoint to at least one angle of 270° or greater.

Similarly to the construction of P_0 we first build \tilde{P}_k by attaching to every non-zero vertical side the rectangle provided by the Lemma 3.4. Notice, that one of the sides of the added rectangle (its base) serves to merge the rectangle to P_k , while the remaining three become sides of \tilde{P}_k .

As before, we claim that none of the added (open) rectangles intersects P_k nor any other added rectangle

However, the argument used to prove the disjointness in the case of P_0 does not work now. Instead we argue as follows.

Suppose that a non-empty open rectangle R_1 attached to the vertical side (v_j, v_{j+1}) intersects P_k (see Figure 7).

Consider first the case when $\text{dist}((v_j, v_{j+1}), R_1 \cap P_k) > 0$. $R_1 \cap \partial P_k$ is nonempty as R_1 contains both points of $\Omega \setminus P_k$ and P_k . Let q is a point of $R_1 \cap \partial P_k$ which is one of the most close to (v_j, v_{j+1}) . Such a point exists as ∂P_k consists of finitely many horizontal and

vertical line segments. The point q belongs to a vertical side of P_k (a non-zero one, as otherwise the angle at q would be 0°). As R_1 is open we may choose q to be not a vertex, and thus being an internal point of some vertical side. Let p is an arbitrary point of (v_j, v_{j+1}) .

As there are no points in $R_1 \cap \overline{P_k}$ which are more close to (v_j, v_{j+1}) than q , the segment (p, q) is entirely contained in $\Omega \setminus \overline{P_k}$. We may consider the curve $(p, q) = \gamma(p, q) = \gamma$. On the other hand, as the open polygon P_k is connected there exists a simple curve $\gamma^* = \gamma^*(p, q)$ which connects p and q inside P_k . As $\gamma \cap \gamma^* = \{p, q\}$ the curve $\gamma \cup \gamma^*$ is a closed simple curve.

By Jordan's theorem the closed simple curve $\gamma \cup \gamma^*$ divides \mathbf{R}^2 in two open domains. One of them is bounded and we denote it $I(\gamma, \gamma^*)$. As $\gamma \cup \gamma^* = \partial I(\gamma, \gamma^*) \subset \Omega$ and Ω is a simply connected domain, we know $I(\gamma, \gamma^*) \subset \Omega$.

As $\gamma \setminus \{p, q\} \subset \Omega \setminus \overline{P_k}$ and $\gamma^* \setminus \{p, q\} \subset P_k$ we have $\partial P_k \cap I(\gamma, \gamma^*) \neq \emptyset$. But ∂P_k is a curve consisting of vertical and horizontal sides of P_k . As $\partial P_k \cap \partial I(\gamma, \gamma^*) = \{p, q\}$ where p and q are internal points of vertical sides, every horizontal side of P_k which intersects $I(\gamma, \gamma^*)$ is contained in it. This means that $\partial P_k \cap I(\gamma, \gamma^*)$ is a curve which contains a non-zero horizontal side of P_k . But then by the $*$ -condition the horizontal side contains a point of $\partial\Omega$, i.e. $\partial\Omega \cup I(\gamma, \gamma^*) \neq \emptyset$. This is a contradiction to $I(\gamma, \gamma^*) \subset \Omega$.

In the case when $d((v_j, v_{j+1}), R_1 \cap P_k) = 0$, the rectangle P_k having only finitely many sides, this means that there exists another side of P_k , say (v_m, v_{m+1}) , such that $(v_j, v_{j+1}) \cap (v_m, v_{m+1})$ is non-empty. If we can take $p \in (v_j, v_{j+1}) \cap (v_m, v_{m+1}) \cap \Omega$, then we may choose a simple closed curve $\gamma \subset P_k \cup \{p\}$, which contains a small horizontal segment with centrum in p . Again, by Jordan's theorem we can consider a bounded open domain $I(\gamma)$, $\partial I(\gamma) = \gamma$, $I(\gamma) \subset \Omega$. As the domain contains part of (v_j, v_{j+1}) it contains part of the (not necessary simple) closed curve ∂P_k . Assuming the natural parametrization ϕ of ∂P_k by \mathbb{T} we see that $\phi^{-1}(p)$ consists of two points which divide \mathbb{T} in two arcs of which one (say τ) parameterizes $I(\gamma) \cap \partial P_k$. It is easy to see that $\phi(\tau)$ contains at least one horizontal side which is either non-zero, or is adjoint to the interior angle of 360° . Again, by the $*$ -condition, this implies $I(\gamma) \cap \partial\Omega \neq \emptyset$ which contradiction proves that $(v_j, v_{j+1}) \cap (v_m, v_{m+1}) \subset \partial\Omega$. But then the rectangle added to the side (v_j, v_{j+1}) should be empty (and so does not have non empty intersection with P_k).

Thus, we have shown that non of the added at this step rectangles can intersect P_k .

Let, now a rectangle R_1 added to a side (v_j, v_{j+1}) intersects a rectangle R_2 added to a side (v_m, v_{m+1}) . Choose $p \in (v_j, v_{j+1})$, $q \in (v_m, v_{m+1})$ and $r \in R_1 \cap R_2$. As we have shown above $R_1, R_2 \in \Omega \setminus \overline{P_k}$, so the curve $\gamma = \gamma(p, q) = [p, r] \cup [r, q]$ satisfies $\gamma \cap (\Omega \setminus \overline{P_k}) = \{p, q\}$. We can choose $\gamma^* = \gamma^*(p, q)$ to be a simple curve connecting p and q inside P_k and the further arguments works exactly as the one in the first case above.

We have, thus, proven that non of the added to P_k rectangles intersects P_k or each other.

Connectedness and simply connectedness of the resulting polygon \tilde{P}_k follows in the same way as in the case of P_0 .

Now we verify that $\tilde{P}_k \in VH(., \frac{1}{2})$. Indeed, if we have added a rectangle to a side of length at least $\frac{1}{2}$, the property (v) of being $VH(., \frac{1}{2})$ is preserved automatically. When we add a rectangle to a side of length less than $\frac{1}{2}$ one of the adjoint angles has to be at least 270° (and as the side is non-zero it has to be exactly 270°). In this case one of the horizontal sides of the added rectangle is a subset of the adjoint horizontal side of the polygon P_k (the side of the rectangle can not be longer than the adjoint side of P_k , as the later contain a point of $\partial\Omega$ while the former does not). So, modifying our previous strategy, we add the half-open horizontal side of the rectangle which is a part of the horizontal side of P_k along with the rectangle (see Figure 8). This modification of the construction does not change the connectedness or simply connectedness of \tilde{P}_k but provides \tilde{P}_k with $VH(., \frac{1}{2})$ property.

When \tilde{P}_k is constructed we proceed in exactly the same manner as in the construction of P_1 to obtain P_{k+1}^* and finally P_{k+1} .

We proceed to construct the sequence of the polygons P_k , till the first moment at which the area of every added rectangle is less than $\frac{1}{4}$. Such step will eventually come, since otherwise the polygons P_k have areas increasing at each step by at least $\frac{1}{4}$ still being subsets of Ω , which would contradict the boundedness of Ω .

3.3.3. Construction of the polygon P Let us consider P_n , the polygon we obtained in the step on which the iterative process described above stopped. We already know that $P_n \in VH(1, \frac{1}{2})$ is connected and simply connected polygon. W.l.o.g. we may assume that the construction stopped at an odd step, so we know that each horizontal side (apart from some of zero length which are adjoined to an angle of 180°) contains a point of $\partial\Omega$. This means that we luck points of $\partial\Omega$ only on vertical sides, or zero-horizontal sides adjoined to an angle of 180° .

Consider the rectangles added in the last step of our construction. Non of them has area greater than $\frac{1}{4}$, which means that all the rectangles added to the horizontal side of length at least $\frac{1}{2}$ has the vertical sides no longer than $\frac{1}{2}$. At the same time the rectangles added to the sides of length less than $\frac{1}{2}$ are no longer than the adjoint by an angle of 270° vertical side of P_{n-1} due to the $*$ -condition, i.e. are not longer than 1.

Let $[v_k, v_l] = \bigcup_{j=k}^{l-1} [v_j, v_{j+1}]$ be the union of any maximal chain of consecutive sides of P_n contained in a single vertical segment. This means that $[v_{k-1}, v_k]$ and $[v_l, v_{l+1}]$ are horizontal sides of P_n of non-zero length and all horizontal sides in the union are of zero length. All vertical side in the union, except possibly $[v_k, v_{k+1}]$ and $[v_{l-1}, v_l]$ which could be vertical sides of the added on the last step rectangles, contain points of $\partial\Omega$, say $q_j \in [v_j, v_{j+1}] \cap \partial\Omega$ for $j \equiv k \pmod{2}$, $k < j < l - 1$. Then we transform the collection of sides of the polygon P_n replacing its sides $[v_k, v_{k+1}], \dots, [v_{l-1}, v_l]$ by the new sides

$$[v_k, q_{k+2}], \quad [q_{k+2}, q_{k+2}], \quad [q_{k+2}, q_{k+4}], \quad \dots, \quad [q_{l-5}, q_{l-3}], \quad [q_{l-3}, q_{l-3}], \quad [q_{l-3}, v_l]$$

(if $l = k + 3$ then we make just one new side $[v_k, v_l]$).

Note that the union of the sides remains unchanged, but every new side contains some point of $\partial\Omega$. Moreover, the new sides has length not exceeding 2. It may happen however that some of the new vertical sides may be shorter then $\frac{1}{2}$. Then we make yet another transformation of the polygon aggregating consecutive short sides together or to the new longer ones. The resulting combination of sides consists of the sides not longer then 3.

Proceeding in that way with all vertical sides we finally get a polygon $P'_n \in VH(3, \frac{1}{2})$ whose every horizontal and vertical side contain a point of $\partial\Omega$, except for some of those which are the vertical sides of the added on the last step rectangles (not connected to a non-zero vertical side of P_{k-1}).

Those later could arise only as a difference between two sides of the added on the last step rectangles, which means they are necessarily adjoined to an angle of 270° (or the vertical side is a zero-length side adjoined to an angle of 180°). For those non-zero sides only we make one last addition of the rectangles provided by the Lemma 3.4 along with the corresponding base and one of the horizontal sides. If it is a zero-length side we shift it till it meets a point of $\partial\Omega$. As every horizontal side contains a point of $\partial\Omega$ the vertical sides won't shift at a distance greater than 1. Thus the length of the modified horizontal sides is not greater than 3.

We see that the polygon obtained as the result of this last adjustment is the desired in the Proposition 3.1 polynomial P : Every side now contains a point of $\partial\Omega$, and the lengthes of the sides are no more than 3, thus $P \in VH(3, \frac{1}{2})$.

3.4. Proof of the proposition 3.2

Let $P \in HV(3, \frac{1}{2})$ and \mathcal{P} be a closed subset of ∂P with nonempty intersection with each side of P .

We will show that, there exist some universal constants δ, Δ, η , such that the polygon P can be cut into η -elementary pieces (A_j) with disjoint interiors, such that $\delta < \text{diam}(A_j) < \Delta$ for every j and such that every point of $\partial P \setminus \mathcal{P}$ belongs to the closure of exactly one of A_j 's. In particular, every open arc of ∂P disjoint with \mathcal{P} intersects the closure of only one A_j .

As a first step, let us consider P_ϵ , the intersection of P and the ϵ -neighborhood of ∂P where $\epsilon < \frac{1}{10}$. Introduce $S_\epsilon = \partial P_\epsilon \cap P$. First we will show that for all $\epsilon \in (0, \frac{1}{10})$ the set S_ϵ is a simple rectifiable curve. Then we will show that S_ϵ is $(10, \frac{\epsilon}{2})$ -Lipschitz, i.e. that shortest sub-curve of S_ϵ connecting two points at Euclidian distance smaller than $\epsilon/2$ is 10-Lipschitz.

Then for $\epsilon = \frac{1}{16}$ and η sufficiently large we will cut P_ϵ in η -elementary pieces of diameter at least $\frac{1}{16}$ and at most 6 in such a way that each point of $\partial P \setminus \mathcal{P}$ belongs to exactly one piece. Finally we will cut $P \setminus P_\epsilon$ in η -elementary pieces of diameter at least $\frac{1}{64}$ and at most $\frac{1}{4}$. Then our task will be accomplished.

3.4.1. Curve S_ϵ We begin from the observation that with natural parametrization the boundary ∂P is a closed piecewise linear curve (not necessarily simple). In the study of S_ϵ it will be convenient for us to remove all zero-length sides τ adjoined to an angle 180° , so we assume that any angle of P is $90^\circ, 270^\circ$ or 360° . Still we may say that $P \in VH(., \frac{1}{2})$, even though the estimate on the lengths of the sides is destroyed by the above removal.

LEMMA 3.5. *S_ϵ contains no isolated points.*

Proof. Observe first that S_ϵ consists of the interior points $z \in P$ with $d(z, \partial P) = \epsilon$.

Let y be an isolated point of S_ϵ . Consider a circle \mathcal{C} with center y and radius r such that neither the circle nor its interior \mathcal{D} contain other points of S_ϵ (r is much less than ϵ). If there are $a, b \in \mathcal{C} \cup \mathcal{D}$ such that $d(a, \partial P) > \epsilon$ and $d(b, \partial P) < \epsilon$ then by the continuity of the distance there is $c \in \mathcal{C} \cup \mathcal{D} \setminus \{y\}$ such that $d(c, \partial P) = \epsilon$. Then $c \in S_\epsilon$ which contradicts the choice of \mathcal{C} .

Therefore, the points of the closed disc should have the distance to ∂P either all greater or all less than ϵ . The former is impossible because $d(y, \partial P) = \epsilon$.

Therefore $d(c, \partial P) < \epsilon$ for all $c \in \mathcal{C}$

Since $d(y, \partial P) = \epsilon$ and ∂P is compact, there exists $p \in \partial P$ such that $d(y, p) = \epsilon$. Let p belongs to a side τ . The distance from a point y to a line segment τ is either the distance to the orthogonal projection of y on τ (with p being the image of y), or the distance to one of the ends of the segment τ (in this case τ and $[y, p]$ form an angle greater than 90°). Let us first consider the later case.

Since $[y, p]$ is a most short way between y and ∂P , we see that $(yp) \subset P$ and the internal angle at p which contains y is at least 270° (see Figure 9-ab). Moreover, $y \in Q(p, P)$.

Let $Q = Q(p, P)$ if the angle at p is 270° , or the half of $Q(p, P)$ containing y if the angle is 360° (see Figure 9b). The side-length of Q is $\frac{1}{2}$. Since $\epsilon < \frac{1}{4}$, circle \mathcal{C}' with center in y and radius ϵ intersects ∂Q in three points - p and, say a and b . As the angle $\angle apb$ inscribed in \mathcal{C}' equals 90° , the segment $[a, b]$ is a diameter of the circle, and y is its middle point. Let z be the point of intersection of \mathcal{C} and the perpendicular bisector of $[a, b]$ outside $\triangle apb$ (remember that $d(y, z)$ is small). Let ℓ be the perpendicular bisector of $[y, z]$ (see Figure 10).

On one hand, if $q \in \partial P$ belongs to the component of $\mathbb{R}^2 \setminus \ell$ which contains y , or to ℓ then $d(q, z) \geq d(q, y) \geq \epsilon$, on the other hand if $q \notin Q$ and q belongs to the component of $\mathbb{R}^2 \setminus \ell$ containing z then $d(q, z) > \epsilon = d(a, y) = d(b, y)$. Since $\partial P \cap Q = \emptyset$ it follows then $d(z, \partial P) \geq \epsilon$. This contradicts to the assumption that $d(z, \partial P) < \epsilon$.

It remains to consider the case when $p \in \partial P$ with $d(p, y) = \epsilon$ is the image of the orthogonal projections of y onto the side τ of P . If p is one of the ends of the side, then the angle at p is at least 180° . Otherwise p is an interior point of some side τ . In both cases $P \in VH(., \frac{1}{2})$ implies

that there exists a rectangle $R \subset P$, such that p is an interior point to the shorter side of R , and the longer side of R is parallel to (y, p) and has length $\frac{1}{2}$ (see Figure 11-ab).

Consider a sequence $\{y_j\} \subset \mathcal{D} \cap (\overline{yp} \setminus [y, p])$ which converges to y . By the choice of \mathcal{D} it should be $d(y_j, \partial P) < \epsilon$, thus there exists a sequence $\{p_j\} \subset \partial P$, such that $d(y_j, p_j) < \epsilon$. Notice, that if $p_j \notin (\overline{yp})_\sigma$, where $\sigma = d(y, \partial R)$. Indeed, $p_j \notin R \subset P$, and if $x \in (\overline{yp})_\sigma \setminus R$, then $d(y_j, x) > \epsilon$.

Let p' is a limit point of $\{p_j\}$. $p' \in \partial P$ and $d(y, p') = \epsilon$. If p' is not the image of an orthogonal projection of y to a side of P we can repeat the argument of the first case. If p' is the image of an orthogonal projection of y to a side of P , then $\overline{yp'}$ is either vertical or horizontal. As $p' \notin (\overline{yp})_\sigma$, it must be $\overline{yp} \perp \overline{yp'}$. As before, there exists a rectangle $R' \subset P$ in which the longer side is parallel to $\overline{yp'}$ and has length $\frac{1}{2}$ (see Figure 12).

Consider now a sequence $\{y'_j\} \subset \mathcal{D} \cap R \cap (\overline{yp'} \setminus [y, p'])$ which converges to y . By the choice of \mathcal{D} there exists $\{p'_j\} \subset \partial P$, such that $d(y'_j, p'_j) < \epsilon$. But, with the same argument as above $p'_j \notin (\overline{yp})_\sigma \cup (\overline{yp'})_{\sigma'}$, where $\sigma' = d(y, \partial R')$. Thus, a limit point of $\{p'_j\}$ (which exists due to compactness of ∂P) can not be the image of orthogonal projection of y on a side of P and we can use the argument of the very first case of the proof. \square

LEMMA 3.6. S_ϵ consists of finitely many arcs of circles of radius ϵ and segments of vertical or horizontal lines.

Proof. Denote the open ϵ -neighborhood of the side τ_k by $N_\epsilon(\tau_k)$ and its boundary by $S_\epsilon(\tau_k)$. We have $S_\epsilon = P \cap (\cup(S_\epsilon(\tau) \setminus (\bigcup_{\tau' \neq \tau} N_\epsilon(\tau'))))$, where the unions run over all sides of P .

The set $S_\epsilon(\tau)$ consists of two intervals parallel to τ_k and two semicircles of radius ϵ . The intersection of $N_\epsilon(\tau)$ with any circle of radius ϵ (not centered at an interior point of τ) or an interval is either an (open) arc of the circle or an open segment of the interval. Thus $S_\epsilon(\tau) \setminus (\bigcup_{\tau' \neq \tau} N_\epsilon(\tau'))$ is a finite union of closed arcs, intervals or points.

By Lemma 3.5 S_ϵ contains no isolated points, so it is a union of finitely many closed arcs or segments. \square

It is convenient further to think not of the polygon P in the plane, but of a covering of \overline{P} by a manifold, which is isometric inside P but do not glue together the points of the curve which bounds P , so that ∂P becomes a simple curve. As S_ϵ is inside P this identification does not affect its topological properties.

Let $\phi(t) : [0, 2n] \rightarrow \partial P$ be the natural parametrization of ∂P , such that each interval $[k, k+1]$ parameterize one side of P . We will consider another parametrization of the curve

$$\tilde{\phi}(t) = \begin{cases} \phi(t-k), & \text{for } t \in [2k, 2k+1]; \\ \phi(k+1), & \text{for } t \in [2k+1, 2k+2]. \end{cases}$$

defined on the interval $[0, 4n]$. (Observe that this is not a simple parametrization even for a simple curve.)

We would like $\tilde{\psi}(t)$ to be a closest to $\tilde{\phi}(t)$ point of S_ϵ with respect to the interior metrics d_P of P , (i.e. $d_P(x, y)$ is the minimal length of a curve connecting x and y inside the polygon P). Notice that in this way the curve $\tilde{\psi}$ is not yet well-defined, because several points of S_ϵ may have the same distance to a point of ∂P . To give a proper definition we have to learn more about the set S_ϵ and the distance of its points to ∂P .

LEMMA 3.7. For any $p \in \partial P$ there exists $z \in S_\epsilon$ with $d_P(p, z) < 4\epsilon$.

Proof. Let τ be a side of P and $p \in \tau$.

Consider first the case $|\tau| > 2\epsilon$. Let $\tau_\epsilon = Q(\tau, P) \cap S_\epsilon(\tau)$. It is clear that τ_ϵ is an interval parallel to τ and of the same length contained in $Q(\tau, P) \subset P$. We have $S_\epsilon \supset \tau_\epsilon \setminus \{z \in \tau_\epsilon : d(z, \partial P) < \epsilon\}$. Since $\partial P \cap Q(\tau, P) = \emptyset$ it is clear that $\tau'_\epsilon = \{z \in \tau_\epsilon : d(z, \partial Q(\tau, P)) > \epsilon\} \subset S_\epsilon$. It is easy to see that τ'_ϵ is a subinterval of τ_ϵ of length $|\tau| - 2\epsilon$ and, in particular, is not empty. The distance from any point of τ to the subinterval τ'_ϵ is at most $\sqrt{2}\epsilon$.

Consider now side $\tau = [a, b]$ of length not exceeding $2\epsilon < \frac{1}{2}$. Since $P \in VH(., \frac{1}{2})$ one of the adjoint angles has to be greater than 90° and, as the angles of the polygon P are 90° , 270° or 360° , is at least 270° . W.l.o.g. we can assume that it is the angle at the vertex b which is 270° or greater.

Consider a point $x \in Q(b, P)$, such that \overline{xb} makes an angle 45° to the vertical and the horizontal direction and $d(x, b) = \sqrt{2}\epsilon$. Since $d(x, \partial P) \geq d(x, \partial Q(b, P)) = \epsilon$ and the distance to ∂P is a continuous function, there exists $z \in [b, x]$ such that $d(z, \partial P) = \epsilon$ and $d(y, \partial P) < \epsilon$ for $y \in (b, z)$. Thus, $z \in S_\epsilon$ and for any $p \in [a, b]$ holds $d(z, p) \leq d(z, b) + d(b, p) \leq \sqrt{2}\epsilon + 2\epsilon < 4\epsilon$. \square

The Lemma 3.7 shows that any point of ∂P is close to S_ϵ . Now we look at the most close points. For $p \in \partial P$ we denote by $m_\epsilon(p)$ the set of all points of S_ϵ which are the closest to p among the points of S_ϵ .

LEMMA 3.8. *If $y \in m_\epsilon(q) \cap m_\epsilon(q^*)$, $y \in S_\epsilon$ for some $q \neq q^*$, then there are two sides $\tau \neq \tau'$ of P , such that $y \in S_\epsilon(\tau) \cap S_\epsilon(\tau')$, and the points $p \in \tau$, $p' \in \tau'$, such that $d(y, p) = d(y, p') = \epsilon$, and $p \neq p'$.*

Proof. Denote by τ the side of P such that $y \in S_\epsilon(\tau)$ (as S_ϵ is a union of subsets of $S_\epsilon(\tau)$ for the sides of P , such a side exists).

Suppose $y \notin S_\epsilon(\tau')$, for any side $\tau' \neq \tau$. Since $y \in S_\epsilon$, we know that y does not belong to ϵ -neighborhood of any side of P . And, as $y \notin S_\epsilon(\tau')$, we know that $d(y, \tau') > \epsilon$ for every side $\tau' \neq \tau$. Thus, there exists small disc $D(y, r)$ such that $S_\epsilon \cap D(y, r) = S_\epsilon(\tau) \cap D(y, r)$, is a 1-smooth curve.

Let p be the closest to y point of τ . Then $d(y, p) = \epsilon$, and $(p, y) \subset P$. By the assumption, there exists another point $q^* \in \partial P$ for which $y \in m_\epsilon(q^*)$. Let γ be the shortest path joining q^* and y inside \overline{P} (here we mean \overline{P} on the universal covering). Consider the closest to y point $q' \in \gamma \cap \partial P$. Then $y \in m_\epsilon(q')$ and $(q', y) \subset P$.

Suppose $p \neq q'$. Since $y \in m_\epsilon(p) \cap m_\epsilon(q')$ and $S_\epsilon \cap D(y, r)$ is a smooth curve passing through y , both (y, p) and (y, q') are orthogonal to the tangent line to $S_\epsilon \cap D(y, r)$ at y . Thus p, q', y are collinear and, since $(p, y) \subset P$ and $(q', y) \subset P$, none of them can contain neither p nor q' . This means that $y \in [p, q']$.

There are two possibilities for placement of \overline{py} with respect to τ :

(i) If p is an interior or endpoint of τ and $\overline{py} \perp \tau$ so that $(y, p) \subset Q(\tau, P)$.

(ii) If p is an endpoint of τ , then the angle at p is greater than 90° and $(p, y) \subset Q(p, P)$

As $q' \in \partial P$, so in the first case, $q' \notin Q(\tau, P)$ and in the second case $q' \notin Q(p, P)$. Then $d(p, q') \geq \frac{1}{2}$ and $d(q', y) \geq \frac{1}{2} - d(y, p) = \frac{1}{2} - \epsilon > 4\epsilon$, which contradicts the Lemma 3.7.

Thus, $p = q' \neq q^*$ and the shortest path from q^* to y passes through p . This can not happen if p is an interior point of the side τ .

If p is an endpoint of τ , then y belongs to one of the semi-circles which constitute $S_\epsilon(\tau)$. Denote the side adjoint to τ at p as τ' .

If the angle at p is 270° , then $S_\epsilon(\tau) \setminus N_\epsilon(\tau')$ contains only half of the semicircle with center at p , which is also contained in $S_\epsilon(\tau')$, thus $y \in S_\epsilon(\tau')$.

If the angle at p is 360° , then the whole semicircle of $S_\epsilon(\tau)$ with center at p is contained in $S_\epsilon(\tau')$, and thus again $y \in S_\epsilon(\tau')$.

Assume now that $y \in S_\epsilon(\tau) \cap S_\epsilon(\tau')$ and $p \in \partial P$ is the only point for which $d(y, p) = \epsilon$. Then $p \in \tau \cap \tau'$ is the common endpoint of τ and τ' and the angle at p is 270° or 360° .

Now either there exists another side τ'' such that $y \in S_\epsilon(\tau'')$ and the lemma follows or, otherwise, there exists some small disc $D(y, r)$ such that $D(y, r) \cap S_\epsilon = D(y, r) \cap S_\epsilon(\tau)$ or $D(y, r) \cap S_\epsilon = D(y, r) \cap S_\epsilon(\tau')$. When the disc exists in the later case we repeat partially the reasoning from above to get a contradiction. Indeed, suppose that there is a point $q^* \neq p$ from ∂P for which $y \in m_\epsilon(q^*)$. Then $d_P(q^*, y)$ is realized by some curve $\gamma \in \overline{P}$, which passes through p .

If the angle at p is 270° then there is $q' \in \gamma \setminus [y, p]$ such that $(y, q') \subset P$ and $[q', y]$ is shorter than the part of curve γ connecting q' to y . This contradicts the choice of γ as a most short path from q^* to y .

If the angle at p is 360° , consider the point y' which is intersection of the line containing τ and the semicircle of $S_\epsilon(\tau)$ centered at p (see Figure 13). By the properties (8) and (5) of $P \in VH(\cdot, \frac{1}{2})$ there is an arc $D(y', r) \cap S_\epsilon(\tau) \subset S_\epsilon$. Then for a suitable chosen $y^* \in D(y', r) \cap S_\epsilon$ and $q' \in \gamma \setminus [y, p]$ holds $(y^*, q') \in P$, and the curve consisting of the part of γ joining q^* with q' and $[y, q']$ is shorter than γ . But then, $d_P(y, q^*) < d_P(y, q^*)$ which contradicts to the assumption $y \in m_\epsilon(p^*)$.

□

Now, we are ready to prove the following important fact.

LEMMA 3.9. *If $z \in m_\epsilon(p) \cap m_\epsilon(q)$ for $p \neq q$ then there is an interval*

$$[t_0, t_1] \subset [0, 4n] / \{0 \sim 4n\}$$

(when we talk about intervals, we assume 0 to be identified with $4n$), such that $m_\epsilon(\tilde{\phi}(t)) = \{z\}$ for all $t \in (t_0, t_1)$. Moreover $d(z, \tilde{\phi}(t_0)) = d(z, \tilde{\phi}(t_1)) = \epsilon$, and $z \notin m_\epsilon(\tilde{\phi}(t))$ for $t \notin [t_0, t_1]$.

Proof. As it was shown in the previous Lemma, there exist $p \in \tau \subset \partial P$ and $p' \in \tau' \subset \partial P$ such that $d(z, p) = d(z, p') = \epsilon$ and $p \neq p'$, $\tau \neq \tau'$.

There are two possibilities for the position of $[z, p]$ with respect to τ (and two for the position of $[z, p']$ with respect to τ'): either $[z, p] \perp \tau$ or p is an end of τ and $[z, p]$ form with τ an angle greater than 90° (in the later case the angle at p is at least 270° , see Figure 14).

Let us first consider the case when both $[z, p] \perp \tau$ and $[z, p'] \perp \tau'$ (and thus $[z, p]$ and $[z, p']$ are either horizontal or vertical). As $P \in VH(\cdot, \frac{1}{2})$, it is clear that $[z, p] \perp [z, p']$. We introduce the coordinate system with z in the origin such that $p = (\epsilon, 0)$ and $p' = (0, \epsilon)$ (this may change the orientation of the plane, in which case we temporary change the numbering of the vertices of the polygon P). We introduce $p^* = (\epsilon, \epsilon)$ so that z, p, p^*, p' are the vertices of a square of the sidelength ϵ which we will call R (see Figure 15). We will show next that one of the two curves into which points p and p' divide ∂P is contained in \overline{R} . We denote this curve γ_R and we will show that $z \in m_\epsilon(p)$ for $p \in \gamma_R$ and $z \notin m_\epsilon(p)$ for $p \in \partial P \setminus \gamma_R$.

Notice that $\tau \perp [z, p]$ and thus is vertical. The point p may be an interior point of τ or its end. In the later case if τ is in the lower half-plane or has zero-length, the angle at p is at least 270° and $p' \in Q(p, P) \subset P$ (see Figure 16-ab) which would be impossible. So in both cases some part of τ is located in the upper half-plane, i.e. $\tau \cap \mathbb{R} \times \mathbb{R}_+ \supsetneq \{p\}$.

The point p^* could not be an interior point of τ because otherwise $[z, p'] \subset Q(\tau, P)$. So the end of τ , which we denote by p_1 , is either inside (p, p^*) or $p_1 = p^*$.

If the angle at p_1 is greater than 90° then $p' \in Q(p_1, P)$. Therefore the angle between τ and the next side of P (which we denote τ_1) is 90° .

If $p_1 \neq p^*$, denote by p_2 the second end of the horizontal side τ_1 (see Figure 17). The side $\tau_1 = [p_1, p_2] \subset \overline{R}$ because all points of (z, p') are the interior points of P and so $\tau_1 \cap (z, p') = \emptyset$.

Hence $|\tau_1| < \epsilon < \frac{1}{2}$ and as $P \in VH(., \frac{1}{2})$ the angle at p_2 is not 90° . Also the angle at p_2 is not 360° , as otherwise $p' \in Q(p_2, P)$. Thus the angle at p_2 is 270° . Denote the second end of the vertical side τ_2 starting at p_2 by p_3 (see Figure 17).

If $p_3 \notin [p^*, p']$ we apply the previous arguments once again. The only possibility is that the angle at p_3 is 90° , and at the next corner, denoted p_4 the angle is 270° , etc. Ultimately, after finitely many (say k) such steps and not later than we reach τ' , we have reached $[p^*, p']$.

This means that $p_k \in [p^*, p']$ (which includes the case $p_1 = p^*$). When it happens the side $[p_{k-1}, p_k]$ is vertical, so the angle at p_k is 90° . By p_{k+1} we denote the next vertex of P . If $p_{k+1} \in (p', p^*)$ then the angle at p_{k+1} is not 90° (because every side shorter than $\frac{1}{2}$ is adjoint to at least one angle greater than 180° due to $P \in VH(., \frac{1}{2})$), and the angle at p_{k+1} can not be 270° or 360° (because otherwise $p' \in Q(p_{k+1}, P)$). Thus, $p' \in [p_k, p_{k+1}]$ and the curve $\gamma_R \subset \partial P$ connecting p and p' is contained in \overline{R} . Moreover, we have established that the angles between the sides composing the curve γ_R alternate between 90° and 270° .

As $P \in VH(., \frac{1}{2})$, the domain bounded by $\gamma_R \cup [z, p] \cup [z, p']$ is interior to P and, thus, no segment connecting z and a point of γ_R intersects ∂P . This means that the interior distance from z to points of γ_R coincide with the Euclidean one.

Now, to finish the consideration of the case when $[z, p] \perp \tau$ and $[z, p'] \perp \tau'$, it remains to show that $m_\epsilon(r) = \{z\}$ for $r \in \gamma_R$ and $z \notin m_\epsilon(r)$ for $r \notin \gamma_R$.

Let $r \in \gamma_R \subset \overline{R}$, $r \neq p, p'$ and $y \in S_\epsilon$, $y \neq z$. As every point of $\overline{R} \setminus \{z, p^*\}$ is at the distance less than ϵ either from p or p' , $y \notin \overline{R}$. This means that there is a curve γ of length no more than $d(z, r)$, which connects r and y inside \overline{P} (we consider \overline{P} in the sense of the internal metric on the covering manifold of P).

Let $q \in \gamma$ be the last point of γ which belongs to the domain bonded by $[z, p] \cup [z, p'] \cup \gamma_R$. Clearly such a point exists and belongs to $[z, p] \cup [z, p']$. W.l.o.g. we may assume that $q \in [z, p]$. Point y has to be outside of the disc $D(p, \epsilon)$. But, if $q \neq p$, since q is on the radius $[p, z]$ of $D(p, \epsilon)$, the most close to q point outside of the disc $D(p, \epsilon)$ is z . Thus γ is strictly longer than $\gamma(r, q) \cup [q, z]$, where $\gamma(r, q)$ is the part of γ connecting r and q . On the other hand, if $q = p$, then $d_P(r, y) = d_P(r, p) + d_P(p, y) \geq d_P(r, p) + d_P(p, z) > d_P(r, z) = d(r, z)$. Thus, in both cases, $d_P(r, y) > d_P(r, z)$, so $y \notin m_\epsilon(r)$, which implies $m_\epsilon(r) = \{z\}$.

We are going to show now that $z \notin m_\epsilon(q)$ for point $q \in \partial P \setminus \gamma_R$. To this end, notice that there exist some small disc $D(z, r)$ such that:

$$D(z, r) \cap S_\epsilon(\tau) \setminus \overline{R} \quad \text{and} \quad D(z, r) \cap S_\epsilon(\tau') \setminus \overline{R}$$

are contained in S_ϵ (depending on whether p and p' are vertices of P or not those sets may be arcs and/or segments, see Figure 18).

We use the following geometrical observation. Suppose that γ is a curve joining z with some point $q \notin \gamma_R$. We will show that there exists another point $z' \in D(z, r) \cap S_\epsilon$ and a curve γ' joining z' with q which is shorter than γ . Observe first, that, as $D(z, \epsilon) \subset P$, the curve $\gamma \cap D(z, \epsilon)$ is a straight segment which is not contained in the Ist or III^d quarters (the first because $q \notin \gamma_R$, the second because $P \in VH(., \frac{1}{2})$).

Now, consider the case when $\gamma \cap D(z, r/2)$ contains a point $q' = (x, y)$ with one of the coordinates, say x , negative. Choose then a point $z' \in (D(z, r) \cap S_\epsilon(\tau')) \setminus \overline{R}$, such that its first coordinate is x . Replace then the part of γ which joints q' with z by the segment $[q', z']$ (call the obtained curve γ'). $[q', z']$ is either a cathetus or a part of a cathetus in a triangle where $[q', z]$ is the hypotenuse (when the part of S_ϵ is a segment, see Figure 19-ab), or $d(z', q') < |y| < |x| < d(q', z)$ (when the part of S_ϵ is an arc, see Figure 20). In both cases the curve γ' is shorter than the curve γ , which implies that z is not a most close to q point of S_ϵ .

It remains to consider a case when (the one of $\gamma \supset [p', z]$ identical). The angle at p can not be 90° as $P \in VH(., \frac{1}{2})$.

If the angle at p is 270° , then $\gamma \supset [p, z]$ is possible only when p is a vertex of P , and γ contains also a part of the non-zero side τ^* adjointed to the side τ . Consider a point q' , $0 < d(q', p) < \epsilon$

such that $q' \in \gamma \cap \tau^*$. Take a point $z' \in S_\epsilon(\tau) \cap D(z^*, r) \setminus \overline{R} \subset S_\epsilon$. The interval $(q', z') \subset P$ is shorter than $d(q', z)$ which shows that γ is not a most short way from q to S_ϵ .

If the angle at p is 360° , denote the point of intersection of the continuation of τ through p with $S_\epsilon(\tau)$ by z^* . As in the proof of Lemma 3.5, we may choose such a small radius r that $D(z^*, r) \cap S_\epsilon(\tau) \subset S_\epsilon$. But then there exists a point $z' \in D(z^*, r) \cap S_\epsilon(\tau)$ with x -coordinate greater than ϵ (see Figure 21). Then $[q', z'] \subset P$ and $d(q', p) + d(p, z) > d(q', z')$, i.e. $d_P(q, z) > d_P(q', z')$ which shows that γ is not a most short way from q to S_ϵ .

Thus, we have shown that $z \notin m_\epsilon(q)$ if $q \notin \gamma_R$.

Summarizing the proof - so far we have shown that if $[z, p] \perp \tau$ and $[z, p] \perp \tau'$, then ∂P can be presented as a union of two curve connecting p and p' . One of them γ_R has the property that $m_\epsilon(r) = \{z\}$ for its interior points r and $z \notin m_\epsilon(q)$ for all $q \in \partial P \setminus \gamma_R$.

Rest of the cases are rather similar to the one described, so we will omit some details. To show the change in the argument we will consider a case when $[z, p] \not\perp \tau$ and $[z, p'] \not\perp \tau'$ (and leave the rest of the cases to the reader).

Recall that if $d(z, p) = \epsilon$ and $[z, p] \not\perp \tau$ then p is an endpoint of τ . As well, p' is an endpoint of τ' and the segments $[p, z]$, and $[p', z]$ make obtuse angles to the corresponding sides τ and τ' (and the angles at p and p' are at least 270°).

As $z \in Q(p, P)$ and $z \in Q(p', P)$, $Q(p, P) \cap Q(p', P) \neq \emptyset$. But then, as $d(p, p') \leq 2\epsilon$, unless $Q(p, P)$ and $Q(p', P)$ is oriented in the same way, either $p \in Q(p', P)$ or $p' \in Q(p, P)$. So, the mutual position of z, p, p', τ , and τ' is the one of indicated in Figure 22 up to a symmetry or rotation. Notice that in all cases the vectors \overrightarrow{zp} and $\overrightarrow{zp'}$ are in the same quadrant, which we by changing the coordinate system can assume to be the I'st quadrant.

Shift, now the origin of the coordinate system in such a way that p and p' are on the positive axes. Wlog we may assume $p = (x, 0), p' = (0, y)$. With this placement of the coordinates z is in the III quadrant. As before, we consider a rectangle $R = P((0, 0), (x, 0), (x, y), (0, y))$ and show that there is a sub-curve of the boundary ∂P , the curve γ_R connecting p and p' , such that $\gamma_R \subset \overline{R}$. Similarly, as above, we observe that $d_P(z, r) = d(z, r)$ for $r \in \gamma_R$.

If $d_P(r, y) \leq d_P(r, z)$ for some $r \in \gamma_R$, we again consider a curve γ which realizes the shortest way between r and y and consider $q \in \gamma \cap ([z, p] \cup [z, p'])$. This leads to a contradiction in exactly the same way as before.

Some difference arises when we look at $\partial P \setminus \gamma_R$. Indeed there may even exist other point p'' for which $d(z, p'') = \epsilon$.

Let us shift the origin to z (preserving the direction of the axes). Both p and p' are in the I quadrant. Consider any other point $p'' \in \partial P$ at the distance ϵ from z . If we repeat the argument for the pair (p, p'') it will lead to the same conclusion, i.e. p'' also is in the I quadrant (if $[z, p''] \perp \tau''$, then p'' has to be on the positive half of one of the axis). Thus, we know that all the points of $q \in \partial P$ such that $d_P(z, q) = \epsilon$ are in the I quadrant. We take as p and p' the pair of such a points for which the angle $\angle(pzp')$ is maximal, and the curve γ_R is the part of ∂P connecting p and p' , as before.

We have already shown that for every interior point of γ_R the point z is the only most close point of S_ϵ .

It remains to show that $z \notin m_\epsilon(q)$ for $q \in \partial P \setminus \gamma_R$. Indeed, if $z \in S_\epsilon(\tau'')$ for some side τ'' , then there exists $p'' \in \tau''$ such that $d(z, p'') = \epsilon$, so $p'' \in \gamma_R$, i.e. $\tau'' \cap \gamma_R \neq \emptyset$. This means that $d(z, N_\epsilon(\tau^*)) > 0$ for any side τ^* disjoint with γ_R , and we again can find so small disc $D(z, r)$ such that $S_\epsilon \supset D(z, r) \cap ((S_\epsilon(\tau) \setminus D(p', \epsilon)) \cup (S_\epsilon(\tau') \cap D(p, \epsilon)))$. Therefore, in the case when neither $[z, p] \perp \tau$, nor $[z, p'] \perp \tau'$ we see that $S_\epsilon \cap D(z, r)$ is a union of two arcs ending in z , which are perpendicular (as smooth curves) to $[z, p]$ and $[z, p']$ respectively (see Figure 23).

Hence, as the last part of the shortest curve connecting q to z inside \overline{P} is a segment, if it has an angle to one of the arcs less than 90° there is a point z' on the arc, and a curve connecting it to q which is shorter. The only way for a most short curve joining q with S_ϵ to end in z is

for its last segment to be contained in the angle $\angle pzp'$. But, as q does not belong to the region bounded by $\gamma_R \cup [z, p] \cup [z, p']$, the shortest way from q to z can not pass inside the region.

This leaves us to consider the cases when $[z, p] \subset \gamma$ or $[z, p'] \subset \gamma$, which can be treated in exactly the same manner as before (both in the cases of the angle 270° and 360°). \square

COROLLARY 3.10. *The interior distances between points of ∂P and the points of $m_\epsilon(p)$ coincide with the Euclidean distances and are realized by a line segment contained in the interior of P , except from one of its ends.*

Proof. Let $p \in \partial P$ and $z \in m_\epsilon(p)$. Suppose that $z \notin m_\epsilon(p')$ for any $p \neq p' \in \partial P$. Since $z \in S_\epsilon$, there exists $p^*(z) = p^* \in \partial P$ for which $d(z, p^*) = \epsilon$, while by the definition of S_ϵ we know $d(p^*, S_\epsilon) = \epsilon$. So $z \in m_\epsilon(p^*)$. Hence $p^* = p$ and $d(p, z) = \epsilon = d(z, \partial P)$, thus $(p, z] \subset P$.

If $z \in m_\epsilon(p) \cap m_\epsilon(p'')$ for two different points of ∂P we use the information from the previous lemma. \square

COROLLARY 3.11. *The segments realizing the shortest distances from the points of ∂P to S_ϵ do not intersect except their ends.*

Proof. Let $[z, p]$, respectively $[z', p']$, realize the shortest distance from p , respectively p' , to S_ϵ .

If $[z, p] \cap [z', p'] = z \neq z'$, then $d(z, p') < d(z', p')$ and $[z', p']$ is not a shortest path from p' to S_ϵ . Similarly it can not be that $[z, p] \cap [z', p'] = z' \neq z$, $[z, p] \cap [z', p'] = p \neq p'$, or $[z, p] \cap [z', p'] = p' \neq p$.

Suppose $[z, p] \cap [z', p'] = q$ for some $q \neq z, p, z', p'$ where $z \in m_\epsilon(p)$ and $z' \in m_\epsilon(p')$. Wlog we may assume $d(q, z) \geq d(q, z')$. Then the length of the curve $\gamma = [p, q] \cup [q, z']$ does not exceed $d(z, p)$ and γ joins p with S_ϵ . As γ is not a straight line and $\gamma \setminus \{p\} \subset P$, we can find a shorter curve connecting p and $z' \in S_\epsilon$ inside P , i.e. $d_P(p, S_\epsilon) < d(p, z)$ and $z \notin m_\epsilon(p)$. This contradiction proves the corollary. \square

LEMMA 3.12. *Let $p \in \partial P$ and $z, z' \in m_\epsilon(p)$ for some $z \neq z'$. Then p is a vertex of P and the angle at p is at least 270° . Moreover, $m_\epsilon(p)$ is a connected arc of a circle of radius ϵ .*

Proof. Assume that $d(p, z) > \epsilon$. Then there exists $p' \in \partial P$, such that $d(p', z) = \epsilon$. This means that there are two different points $p, p' \in \partial P$ for which $z \in m_\epsilon(p) \cap m_\epsilon(p')$. Applying Lemma 3.9 we get $p \in \tilde{\phi}[t_0, t_1]$ and $p \neq \tilde{\phi}(t_0), \tilde{\phi}(t_1)$ because $d(p, z) > \epsilon$. But then $S_\epsilon(p) = \{z\}$. This contradiction shows that $d(p, z) = d(p, z') = \epsilon$.

If p were an interior point of a side $\tau \subset \partial P$, then there exist exactly two points outside $N_\epsilon(\tau)$ at the distance ϵ from p . Only one of them can be connected to p by a segment described in the Corollary 3.10. Any other point of S_ϵ will have the distance to p greater than ϵ . This means that there is only one closest to p point of S_ϵ and the contradiction shows that p has to be an endpoint of a side, i.e. a vertex.

Let p is an endpoint point of a side τ . We already know that $d(p, S_\epsilon) = \epsilon$. This means that $m_\epsilon(p)$ is contained in the semicircle with the centra in p which is a part of $S_\epsilon(\tau)$. Let us denote this semicircle by γ_p .

If the angle at p is 90° the line segment which connects p to $z \in S_\epsilon(p)$ by Corollary 3.10 forms an acute angle with both sides of P ending in p . Thus there are points of ∂P which are closer to z than p . This contradicts $d(z, p) = \epsilon$. Thus the angle at p is at least 270° .

Since $S_\epsilon = (\cup (S_\epsilon(\tau') \setminus (\bigcup_{\tau'' \neq \tau'} N_\epsilon(\tau''))))$, where the first union is taken over all sides of P , we have $m_\epsilon(p) = \gamma_p \setminus (\bigcup_{\tau'' \neq \tau'} N_\epsilon(\tau'')) = \gamma_p \setminus (\partial P + D(0, \epsilon))$. Let $\mathcal{Q} \subset \partial P$ be a countable dense subset. Then $\partial P + D(0, \epsilon) = \bigcup_{q \in \mathcal{Q}} D(q, \epsilon)$. We will show that $\gamma_p \setminus \bigcup_{q \in \mathcal{Q}} D(q, \epsilon)$ is a connected arc.

Let us first consider the case when the angle at p is 270° . Denote the other side of P ending in p by ν (as we consider the covering of P this side is uniquely defined). W.l.o.g. we may assume that p is in the origin of cartesian coordinates, τ goes along the positive x -axis, and ν goes along the positive y -axis. Then $\gamma_p \setminus N_\epsilon(\nu)$ is contained in a quart-circle C in the third quadrant. What part of this quart-circle belongs to S_ϵ depends from what is left after we take away all the discs $D(q, \epsilon)$, $q \in \mathcal{Q}$. Notice that if $q \in \partial P$ is a point of the third quadrant, then $q \notin Q(p, P)$ and $D(q, \epsilon) \cap \gamma_p = \emptyset$ (by property (vii) of $P \in VH(\cdot, \frac{1}{2})$). On the other hand, if the point q is outside of the third quadrant then $D(q, \epsilon) \cap \gamma_p$ is a connected arc containing one of ends of C . This means that the union of all intersections $D(q, \epsilon) \cap \gamma_p$ forms two arcs containing the corresponding ends of γ_p and what left after removing of these arcs is a connected arc.

If the angle at p is 360° we may w.l.o.g. assume that the sides go along the positive part of the x -axis. Then similar reasoning can be applied with replacing the quart-circle by the semicircle for γ_p and the third quadrant by the left half-plane. \square

We are going to define now the continuous function $\tilde{\psi} : [0, 4n] \rightarrow S_\epsilon$, such that $\tilde{\psi}(t) \in m_\epsilon(\tilde{\phi}(t))$. This condition defines $\tilde{\psi}$ uniquely on the intervals $(2k, 2k+1)$, because, by Lemma 3.12, all points of the sides of P except vertices have exactly one closest point of S_ϵ .

LEMMA 3.13. *The function $\tilde{\psi}$ is continuous on $(2k, 2k+1)$.*

Proof. Let and $t_j \in (2k, 2k+1)$, and $\lim t_j = t^*$, $t^* \in (2k, 2k+1)$. Then $\tilde{\phi}(t_j) \rightarrow \tilde{\phi}(t^*)$, and so $d(\tilde{\phi}(t_j), \tilde{\psi}(t_j)) = d(\tilde{\phi}(t_j), S_\epsilon) \rightarrow d(\tilde{\phi}(t^*), S_\epsilon)$. Consider a limit point z of the sequence $\tilde{\psi}(t_j)$. For this point holds $d(\tilde{\phi}(t^*), z) = \lim d(\tilde{\phi}(t_j), \tilde{\psi}(t_j)) = d(\tilde{\phi}(t^*), S_\epsilon)$ and $z \in S_\epsilon$, as the set S_ϵ is closed.

As $\tilde{\psi}(t^*)$ is the unique point of S_ϵ , which is at the distance $d(\tilde{\psi}(t^*), S_\epsilon)$ of $\tilde{\psi}(t^*)$, it has to be $z = \tilde{\psi}(t^*)$. Thus, the sequence $\tilde{\psi}(t_j)$ of points of the compact set S_ϵ has a unique limit point $\tilde{\psi}(t^*)$, i.e. $\tilde{\psi}$ is continuous in t^* . \square

Let us now consider the vertices of P . If the angle at a vertex p is less than 270° , then the Lemma 3.12 shows that the closest to p point of S_ϵ is unique, and the same proof as in Lemma 3.13 shows that setting $\tilde{\psi}(t)$ for $t \in \tilde{\phi}^{-1}(p)$ to the unique point of $m_\epsilon(p)$ we get the function $\tilde{\psi}$ to be continuous in a neighborhood of $\tilde{\phi}^{-1}(p)$.

It remains to consider the cases when the angle at p is 270° or 360° . In both cases the set $m_\epsilon(p)$ is either a point or a (connected) arc by Lemma 3.12. If the arc has degenerated to a point, then the above argument show that setting $\tilde{\psi}$ to that point for $\tilde{\phi}^{-1}(p)$ will define $\tilde{\psi}$ continuously on $\tilde{\phi}^{-1}(p)$ and in its neighborhood. Non-degenerated case is described by the following lemma.

LEMMA 3.14. *If $p = \tilde{\phi}([2k+1, 2k+2])$ is a corner of P with the intern angle at least 270° such that $m_\epsilon(p)$ is an arc γ_p , then $\lim_{t \rightarrow (2k+1)^-} \psi(t)$ and $\lim_{t \rightarrow (2k+2)^+} \psi(t)$ exist and are the endpoints of the arc γ_p . $\lim_{t \rightarrow (2k+1)^-} \psi(t) \neq \lim_{t \rightarrow (2k+2)^+} \psi(t)$.*

Proof. Let $p = \tilde{\phi}([2k+1, 2k+2])$. Consider first the case of 270° .

We place the origin of the coordinate system to p , and choose the direction of axes so that the intern angle at p contains II, III and IV quadrants, and so that the side $\tilde{\phi}([2k, 2k+1])$ goes along positive x -axis and $\tilde{\phi}([2k+2, 2k+3])$ - along positive y -axis (see Figure 24). Let the upper end of the curve $\gamma_p = m_\epsilon(p)$ is $r = (x_r, y_r)$. Consider a point $q = \tilde{\phi}(t)$, $t \in (2k+2, 2k+3)$. The point q is on the positive y -axis, and has coordinates $q = (0, y_q)$. Notice that $y_r \leq 0$ and $y_q > 0$.

As q is not a vertex, $\tilde{\psi}(t)$ is the unique point of S_ϵ most close to q , and thus is at the distance at most $d(q, r)$ from q . At the same time, it should be in S_ϵ , and thus $\tilde{\psi}(t) \notin D(p, \epsilon)$. This means that $\tilde{\psi}(t)$ has y -coordinate not less than the one of r , as any point of $(\mathbb{R} \times (y_r, -\infty)) \setminus D(p, \epsilon)$ has distance to q greater than $d(y, q)$.

Now, consider $(t_n) \subset (2k+2, 2k+3)$, such that $t_n \rightarrow 2k+2$. Then $\tilde{\phi}(t_n) \rightarrow p$ along y -axis. $d(\tilde{\psi}(t_n), \tilde{\phi}(t_n)) \leq d(r, \tilde{\phi}(t_n))$. For any limit point z of $(\tilde{\psi}(t_n))$ holds, thus, $d(p, z) \leq \epsilon$. As $z \in S_\epsilon$ implies $d(z, p) \geq \epsilon$, we see that $d(p, z) = \epsilon$.

By Lemma 3.12, $z \in \gamma_p$. At the same time the y -coordinate of any limit point should be no less than y -coordinate of r . This means that $z = r$. As the sequence is bounded, this means that the limit $\lim \tilde{\psi}(t_n)$ exists and is r . The side $\tilde{\phi}([2k, 2k+1])$ can be dealt with in the same manner.

Now, let the angle is 360° . Then we can place the coordinate system so that the origin is at p and the sides attached to it go along the positive y -axis. Consider again $\tilde{\phi}(t_n) \rightarrow p$, such that $t_n \in (2k+2, 2k+3)$ (and so belong to the side which is "attached" to the interior of P in the second quadrant). By Corollary 3.10, the interior distance between $\tilde{\phi}(t_n)$ and $\tilde{\psi}(t_n)$ is the same as the Euclidean distance. This means that all $\tilde{\psi}(t_n)$ are in the left half-plane and so are the limit points of the sequence. As the bounded sequence $(\tilde{\psi}(t_n))$ has at least one limit point and this point belongs to the arc $\gamma_p = m_\epsilon(p)$, the arc has some points in the closed left half-plane. Consider the highest of those points and repeat the argument we have used for the angle of 270° . The side $\tilde{\phi}([2k, 2k+1])$ can be dealt with in the same manner. \square

Taking into the account Lemmas 3.12 and 3.14 we can use $\tilde{\phi}^{-1}(p)$ to parameterize the arc $m_\epsilon(p)$ in such a way that $\tilde{\psi}$ is continuous in all points of $\tilde{\phi}^{-1}(p)$ (including the ends). This completes the construction of a continuous function $\tilde{\psi} : [0, 4n] \mapsto S_\epsilon$. As every point of S_ϵ is at the distance ϵ of some $p \in \partial P$, it is in $m_\epsilon(p)$ for some $p \in \partial P$. Thus $\tilde{\psi}([0, 4n]) = S_\epsilon$, which shows that S_ϵ is a curve, may be not simple (notice that we keep the assumption that $\tilde{\psi}(0) = \tilde{\psi}(4n)$).

In order to show that the curve S_ϵ is simple, it is enough to show that for any $q \in S_\epsilon$ the set $\tilde{\psi}^{-1}(q)$ is a closed interval (or a point). Then, by factorizing $[0, 4n]$ by those intervals we obtain a simple parametrization of S_ϵ .

By the construction, $\tilde{\psi}^{-1}(q)$ can be more than one point only in two cases: there are $p, p' \in \partial P$ such that $q \in m_\epsilon(p)$ and $q \in m_\epsilon(p')$, or there is a vertex p such that $m_\epsilon(p) = \{q\}$ (and $q \notin m_\epsilon(p')$ for $p \neq p' \in \partial P$). In the first case the Lemma 3.9 gives us the desired property, and the second case is trivial.

Now, when we know that S_ϵ is a simple curve, Lemma 3.6 shows that the curve S_ϵ is rectifiable.

3.4.2. Lipschitz property of S_ϵ

DEFINITION 3. We say that a simple curve γ is η -Lipschitz, if a distance between any two of its points measured along the curve (if the curve is closed we take the shortest of the two pathes along γ connecting the points) is at most η times the Euclidean distance between the points. We say that a curve is μ -locally η -Lipschitz if the above holds for points at Euclidean distance closer then μ .

Note that an η -Lipschitz curve is necessarily simple.

In this section we will show that S_ϵ is not only a simple closed curve (as it was shown above), but also an $\frac{\epsilon}{4}$ -local η -Lipschitz curve for some universal constant $\eta > 0$.

LEMMA 3.15. *Given $r > 0$ such that $r + \epsilon < 1/10$, for every point $x \in S_\epsilon$ there exists a point $y \in S_{\epsilon+r}$, such that $d(x, y) < 2r$.*

Proof. Let $x \in S_\epsilon$, $p \in \partial P$ satisfies $d(x, p) = \epsilon$ and let $m_\epsilon(p) = \{x\}$. Consider $y \in m_{\epsilon+r}(p)$.

If $\{y\} \notin m_{\epsilon+r}(q)$ for $q \neq p$ then (as it has been shown) $d(p, y) = \epsilon + r$. By continuity of the distance to ∂P , there exists a point $x' \in S_\epsilon \cap (p, y)$. Clearly $d(x', \partial P) \leq d(x', p)$, so $d(x', p) \geq \epsilon$ and $d(x', y) \leq r$. Then $d(y, \partial P) \leq d(x', \partial P) + d(x', y) \leq \epsilon + r$ and $y \in S_{\epsilon+r}$ implies that all the inequalities in the estimates must be equalities. Thus $d(x', p) = \epsilon$ and $x' = x$. But then $d(x', y) = r$.

Let $y \in m_{\epsilon+r}(q)$ for some $q \neq p$. Then by lemma 3.9 there are $t_0, t_1 \in [0, 4n]$ such that $p \in \tilde{\phi}([t_0, t_1])$, $y \in m_{\epsilon+r}(z)$ for all $z \in \tilde{\phi}([t_0, t_1])$ and $d(y, t_0) = d(y, t_1) = \epsilon + r$. Let $p_0 = \tilde{\phi}(t_0)$ and $p_1 = \tilde{\phi}(t_1)$. Arguing as above, we derive that $S_\epsilon \cap (y, p_0) = \{x_0\}$, $S_\epsilon \cap (y, p_1) = \{x_1\}$, and $d(y, x_0) = d(y, x_1) = r$. By continuity of the distance, $S_\epsilon \cap (y, p) \neq \emptyset$. The interval (y, p) is contained in the polygon Q bounded by $[y, p_0]$, $\tilde{\phi}([t_0, t_1])$, and $[y, p_1]$. Hence $S_\epsilon \cap Q \neq \emptyset$. As we already know that S_ϵ is a simple curve, it follows that $S_\epsilon \cap Q$ is a curve γ connecting x_0 and x_1 (and has no other points of intersection with ∂Q).

By Lemma 3.11 the segment $[p, x]$ does not intersect neither $[p_0, x_0]$ nor $[p_1, x_1]$, unless $x = x_0$, $x = x_1$, $p = p_0$, or $p = p_1$, because they all are segments connecting points of ∂P to its closest points of S_ϵ . If $x = x_0$ or $x = x_1$ then the distance from $d(x, y) = r$, and as $m_\epsilon(p) = \{x\}$ the case $p = p_0$ implies $x = x_0$ and $p = p_1$ implies $x = x_1$.

Also $[p, x] \cap [y, x_0] = \emptyset$ and $[p, x] \cap [y, x_1] = \emptyset$ because $d(q, \partial P) \leq \epsilon$ for every $q \in [x, p]$. Since, obviously, $[x, p] \cap \tilde{\phi}([t_0, t_1]) = \emptyset$, it follows that $[x, p] \cap \partial Q = \emptyset$ and $x \in \gamma = \overline{Q} \cap S_\epsilon$ (as soon as $p \neq p_0$, $p \neq p_1$). The proof of the Lemma 3.9 describes the shape of $\partial P \cap \overline{Q}$, and this yields that $S_\epsilon \cap Q$ is contained in the part of Q cut by the horizontal/vertical lines passing through x_0 and x_1 (see Figure 25). That parts of Q have diameters less than $2r$, so $d(x, y) < 2r$. Notice that the argument above doesn't use that $m_\epsilon(p) = \{x\}$.

Finally, let $m_\epsilon(p) \neq \{x\}$ (i.e. $m_\epsilon(p)$ contain points other than x) and $y \notin m_{\epsilon+r}(q)$ for $q \neq p$. Then, Lemma 3.12 implies that p is a vertex of P . Denote the connected simple arc $m_\epsilon(p)$ by $\gamma_p \subset S_\epsilon$. The set $m_{\epsilon+r}(p)$ may be an arc or a point, of which we will consider only the first as the later is only degenerated case of the former.

Let y' and y'' are the endpoints of the arc $m_{\epsilon+r}(p)$. Denote the sides adjoint to p as τ' and τ'' . The points y' and y'' may or may not be endpoints of the arc $\overline{D}(p, \epsilon + r) \setminus (N_{\epsilon+r}(\tau') \cup N_{\epsilon+r}(\tau''))$ (see Figure 26). We consider the case when y'' is and y' is not an endpoint, as all the other cases can be treated by a similar argument.

As y' is not an end of $\overline{D}(p, \epsilon + r) \setminus (N_{\epsilon+r}(\tau') \cup N_{\epsilon+r}(\tau''))$, there exists $p' \in \partial P \setminus (\tau \cup \tau')$ such that $d(y', p') = \epsilon + r$. Lemma 3.9 implies that in a subcurve γ_p of ∂P connecting p and p' all points have y' as the closest point of $S_{\epsilon+r}$ and we can repeat the argument of the previous case for all points of S_ϵ in the region Q bounded by γ_p , $[y', p]$, and $[y', p']$ (see Figure 27). On the other hand, for the region Q^* , bounded by $m_{\epsilon+r}(p)$, $[y', p]$, and $[y'', p]$, the set $S_\epsilon \cap Q^*$ is obviously an arc of the circle of radius ϵ , thus every point of it is within distance r from $m_{\epsilon+r}(p) \subset S_{\epsilon+r}$ (see Figure 28).

To complete the proof of this case it suffice that we show that $m_\epsilon(p) \subset Q \cup Q^*$. Indeed denote the point of $[y', p'] \cap S_\epsilon$ as x' and the point of $[y'', p] \cap S_\epsilon$ as x'' . The arc $m_\epsilon(p)$ can not extend further than x'' , as it should be a subset of $\overline{D}(p, \epsilon) \setminus (N_\epsilon(\tau') \cup N_\epsilon(\tau''))$. Also, as by triangle inequality $d(y', x') + d(x', p) > \epsilon + r$, and $d(y', x') = r$, we know that $d(x', p) > \epsilon$,

i.e. $x' \notin m_\epsilon(p)$. By continuity of the distance applied to $[y', p]$, we see that there are points of $m_\epsilon(p)$ inside the region $Q \cup Q^*$, and the curve $m_\epsilon(p)$ does not intersect $\partial(Q \cup Q^*)$, except probably by its end point. This means that $m_\epsilon(p) \subset Q \cup Q^*$, which completes the proof of our last case. \square

LEMMA 3.16. *For any $x, y \in S_\epsilon$, such that $d(x, y) = r < \frac{1}{2}\epsilon$, where $r + \epsilon < 1/10$, one of the two components of $S_\epsilon \setminus \{x, y\}$ is contained in a ball of radius not greater than $3r$.*

Proof. Let $p_x, p_y \in \partial P$ satisfy $d(x, p_x) = d(y, p_y) = \epsilon$. Then all the points of $[x, p_x]$ and $[y, p_y]$ have distance to ∂P not greater than ϵ .

By the triangle inequality, $\epsilon - r/2 \leq d(z, \partial P) \leq \epsilon + r/2$ for every $z \in [x, y]$. The curve $\gamma = [p_x, x] \cup [x, y] \cup [y, p_y]$ divides P on two parts P_1 and P_2 . One of them, say P_1 , is disjoint with $S_{\epsilon + \frac{\sqrt{3}}{2}r}$ because $\gamma \cap S_{\epsilon + \frac{\sqrt{3}}{2}r} = \emptyset$.

Let $\tilde{\gamma} = \partial P \cap \overline{P_1}$. We have already proven that $\gamma' = \tilde{\psi}(\tilde{\phi}^{-1}(\tilde{\gamma}))$ is a simple curve (we consider $\tilde{\gamma}$ on the covering of P). Clearly $x, y \in \gamma'$. Denote by γ_{xy} the part of γ' connecting x and y . Then γ_{xy} is a sub-curve of the simple curve S_ϵ . We will show first that $d(z, [x, y]) \leq 2r$ for every $z \in \gamma_{xy}$.

Indeed, if $z \in \gamma_{xy} \setminus P_1$ we consider $p \in \partial P$ such that $d(p, z) = \epsilon$ (and so $(p, z) \subset P$). Then there is t such that $\phi(t) = p$ and $\psi(t) = z$. By the choice of γ_{xy} , $t \in \phi^{-1}(\partial P \cap \partial P_1)$ so $p \in \partial P \cap \partial P_1 \setminus \{p_x, p_y\}$. The segment (z, p) intersects thus γ , but by Corollary 3.11 it can not intersect neither $[x, p_x]$ nor $[y, p_y]$. Then there for $s \in (x, y) \cap (z, p)$ holds $d(z, s) = d(z, p) - d(s, p) \leq \epsilon - (\epsilon - r/2) = r/2$. Hence $\gamma_{xy} \setminus P_1 \subset ([x, y])_{\frac{1}{2}r}$.

If, on the other hand, $z \in P_1 \cap \gamma_{xy}$ then, since $z \in S_\epsilon$, it satisfies, by the Lemma 3.15, $d(z, S_{\epsilon + \frac{\sqrt{3}}{2}r}) \leq \sqrt{3}r$. But $d(z, P \setminus P_1) > \sqrt{3}r$ for every $z \in P_1 \setminus (D(p_x, \epsilon) \cup D(p_y, \epsilon) \cup ([x, y])_{2r})$. We see that $\gamma_{xy} \cap P_1 \subset ([x, y])_{2r}$, as $S_\epsilon \cap (D(p_x, \epsilon) \cup D(p_y, \epsilon)) = \emptyset$.

In all cases we get $\gamma_{xy} \subset [x, y] + D(0, 2r) \subset D(\frac{x+y}{2}, 3r)$. \square

LEMMA 3.17. *If $r < \frac{1}{4}\epsilon$ then the length of intersection $S_\epsilon \cap D(a, r)$, does not exceed $20r$.*

Proof. Recall that S_ϵ consists of horizontal and vertical segments, and arcs of circles of radius ϵ .

Consider first only the part V of $S_\epsilon \cap D(a, r)$ consisting of vertical segments. It follows from the construction that if $z \in V$ is an interior point of a vertical segment then there exist a horizontal segment $[z, p]$ of length ϵ such that $p \in \partial P$ and $d(s, \partial P) < \epsilon$ for every $s \in (z, p)$. Hence $(z, p) \cap S_\epsilon = \emptyset$. This means that any horizontal line intersects V in at most two points. Hence the total length of V does not exceed twice the length of the projection of V onto the vertical axis which is bounded by $2r$ because $V \subset D(a, r)$. That gives an estimate $4r$ for the length of V . The case of the horizontal counterpart is similar.

The part $C \subset S_\epsilon \cap D(a, r)$ consisting of circular arcs decomposes onto parts C_1, C_2, C_3 and C_4 contained respectively subarcs of the arcs in the first, second, third, and forth quarters with respect to the coordinate system centered in the center of the corresponding circle. We project $C_1 \cup C_3$ orthogonally onto the line $x + y = 0$. For every point z of an arc in $C_1 \cup C_3$ which centers in p , there exists an open segment parallel to $x - y = 0$ which has an end in z , and length $\sqrt{2}\epsilon$ such that it lies in $D(p, \epsilon)$ and, thus, does not intersect S_ϵ . An argument similar to the one above proves that every point of the line is the image of at most two points of $C_1 \cup C_3$. Since the projection of the arcs is $\sqrt{2}$ bi-Lipschitz, the total length of $C_1 \cup C_3$ is bounded by $4\sqrt{2}r$. The same argument works for $C_2 \cup C_4$.

All together, the parts of $S_\epsilon \cap D(a, r)$ have total length bounded by $(8 + 8\sqrt{2})r < 20r$. \square

Combining the above two lemmas we get that S_ϵ is $\frac{1}{12}\epsilon$ -locally η -Lipschitz, for $\eta = 60$.

3.4.3. Decomposition of P Before we conduct the decomposition of P we need the following simple lemma.

LEMMA 3.18. *When we consider the polygon as a manifold with respect to its internal metric, then ∂P (as boundary with respect to the internal for P metrics) is $\epsilon/4$ -locally $\sqrt{2}$ -Lipschitz.*

Proof. Let $x, y \in \partial P$ and $d_P(x, y) < \epsilon/4$. Consider a curve which realizes this distance $\gamma \subset \overline{P}$. This curve consists of points of ∂P . If two points from the same side of P belong to γ , then the whole interval between them is a subinterval of the side, as this is the shortest path between the two points. Thus, at most two ends of the open intervals can belong to the same side of P . As P has finitely many sides, this means γ contains at most finitely many open intervals of P and the rest is finitely many subintervals or points of ∂P . It suffices thus show that every open interval of $\gamma \cap P$ (of length less than $\epsilon/4 < \frac{1}{2}$) can be replaced by a curve in ∂P with the same ends, and the length not more than $\sqrt{2}$ as large. This argument is very similar to the one in the proof of Lemma 3.9, and is left to the reader. \square

To decompose P we consider two curves S_ϵ and $S_{2\epsilon}$ where $\epsilon = \frac{1}{32}$ (remember that P is re-scaled to keep $d = 1$).

Since S_ϵ and $S_{2\epsilon}$ are simple curves, each of them is a boundary of a connected simply connected domain. Thus P is divided onto three parts: Ω_1 bounded by ∂P and S_ϵ ; Ω_2 bounded by S_ϵ and $S_{2\epsilon}$; Ω_3 bounded by $S_{2\epsilon}$.

We will decompose each part separately, though decompositions of Ω_2 and Ω_3 will blend on the boundary.

We begin with Ω_1 . In the description bellow we use the notion of cut. By a *cut* we mean a segment or a union of two segments with common ends such that one of its endpoints is contained in $\partial P \cap \partial\Omega$ and another in S_ϵ . Connected components of Ω_1 with all the cuts removed are elements of the desired decomposition (and due to the first condition every connected component of $\Omega \setminus P$ is attached to only one element of decomposition of Ω_1). We call a cut κ *regular* provided that for any pair of points x, y such that $x \in \kappa, y \in \partial P \cup S_\epsilon$, the Euclidian distance $d(x, y)$ is not more than $\tilde{\eta}$ times as short ($\tilde{\eta} > 0$ is a universal constant) as a most short curve in $\kappa \cup \partial P \cup S_\epsilon$ connecting x and y , as soon as the internal for P distance between x and y is at most $\epsilon/36$.

First we describe cuts with one end on a side τ adjoint to two angles of 90° (see Figure 29). Note that by condition (v) of $P \in VH(\cdot, \frac{1}{2})$ every such side has length at least $\frac{1}{2}$.

Since $S_\epsilon \supset S_\epsilon(\tau) \setminus (\partial P + D(0, \epsilon))$, by the condition (v) of $P \in VH(\cdot, \frac{1}{2})$ there is an interval $I_\tau \subset S_\epsilon$, which is parallel to τ such that $d(\tau, I_\tau) = \epsilon$, and the image of I_ϵ projected orthogonally on τ covers all points of τ with distance from its endpoints greater then ϵ .

Before we removed the sides of length zero, in order to study S_ϵ , the polygon P was in the class $VH(3, \frac{1}{2})$. Thus, there are no subinterval of τ of length greater than 6 disjoint with $\partial\Omega$. If $|\tau| \geq 6 + 4\epsilon$, than there is $p \in \tau \cup \partial\Omega$ and a point $z \in I_\tau$ such that $[p, z] \perp \tau$ and $d(p, \partial\tau) > 2\epsilon$. Let $[p, z]$ be a cut. If $|\tau| > 6k + 4\epsilon$ for some integer k , we make k such cuts dividing τ into pieces of length greater than ϵ and less than $6 + 4\epsilon$ (see Figure 29). It is clear that such cuts are regular.

If the length of τ is less then $6 + 4\epsilon$ there still exists $p \in \tau \cap \partial\Omega$. Let z is middle of the interval I_τ . We show that the cut $[p, z]$ is regular. As z is the middle of I_τ , $|I_\tau| > \frac{1}{2} - 2\epsilon = 14\epsilon$ it is clear that if $x \in S_\epsilon$ and $d(x, [p, z]) < \epsilon/36$ then $x \in I_\tau$. As $|\tau| < 196\epsilon$ and $d(I_\tau, \tau) = \epsilon$, it is clear that

there exists a universal constant $\tilde{\eta}$ such that $d(x, z) + d(z, y) < \eta d(x, y)$, for $x \in S_\epsilon, y \in [p, z]$, when $d_P(x, y) \leq \epsilon/36$.

Notice that either $d(p, \partial\tau) > \epsilon/36$ or the angle between $[p, z]$ and τ is less than $\frac{\pi}{3}$ (see Figure 30). Let $x \in \partial P$, $y \in [p, z]$ and $d_P(x, y) \leq \epsilon/36$. If $d(p, \partial\tau) > \epsilon/36$ then, as $Q(\tau, P) \cap \partial P = \emptyset$, it must be that $x \in \tau$, and the case follows to the same argument as the previous one.

On the other hand if $x \notin \tau$ then it has to be outside of $Q(\tau, P) \cup \tau$. In such a case (see Figure 31) $d(p, \partial\tau) < \epsilon/36$ and $d_P(y, x) \geq \frac{2}{\sqrt{3}}d_P(p, y)$, so $d_P(p, x) < d_P(p, y) + d_P(y, x) < \epsilon/12$. By Lemma 3.18 there exists $\gamma_1(p, x) \subset \partial P$ such that $|\gamma_1| \leq \sqrt{2}d_P(p, x) \leq \sqrt{2}(\frac{\sqrt{3}}{2} + 1)d_P(y, x) < 6d_P(y, x)$. Thus, the curve $[y, p] \cup \gamma_1$ connects x and y inside $\partial P \cup [p, z]$, and $|[y, p] \cup \gamma_1| \leq 8d_P(x, y)$, which completes the proof of the regularity of the cup $[p, z]$.

Consider now a vertex p adjoint to an angle of 360° . As a matter of fact there are two non-zero and one zero length sides ending in p . Because of the zero length side, we know that $p \in \partial\Omega$. We make a cut $[p, z]$ where $z \in S_\epsilon$ to be a continuation an the non-zero sides with an end in p (see Figure 32). As $z \in m_\epsilon(p)$ for any $y \in [p, z]$ holds $d(y, z) = d(y, S_\epsilon)$. Thus, if $d(y, x) < \epsilon/36$, $x \in S_\epsilon$, then $d(y, z) < d(y, x)$ and by the triangle inequality $d(z, x) < 2d(y, x)$. Applying $\epsilon/12$ -local η -Lipschitz property of S_ϵ we can find a curve $\gamma_1(z, x) \subset S_\epsilon$ such that $|\gamma_1| < 2\eta d(x, y)$. The curve $\gamma(y, x) = [y, z] \cup \gamma_1$ has thus length at most $(2\eta + 1)d(x, y)$, which proves regularity of the cut $[p, z]$ on that end of the cut. On the other hand, $[p, z]$ is orthogonal to a side of $Q(p, P)$, thus $d(y, p) = d(y, \partial P)$ for every $y \in [p, z]$ and a similar argument show regularity of the cut $[p, z]$ on its other end.

Let us consider a side τ adjoint to two angles of 270° . By the conditions (v) and (vii) of $P \in VH(\cdot, \frac{1}{2})$ we conclude that there is $I_\tau \subset S_\epsilon$, which is parallel to τ and its orthogonal projection that covers all τ (see Figure 33). Thus we take $p \in \tau \cap \partial\Omega$ and make the cut $[p, z] \perp \tau$. The same argument as before assure that the cut is regular, though we have to modify the definition of a regular cut so that the estimate of the distance along the curve via Euclidian holds only till the next cut.

Observe now that after performing all the above cuts Ω_1 is divided into such pieces that every side of ∂P which belongs entirely to the closure of one of them, say A , is adjoint to exactly one angle of 90° and one of 270° . This means that curves $\bar{A} \cap \partial P$ are $\sqrt{2}$ -Lipschitz not only with respect to interior metrics of P but also with respect to Euclidean metrics.

Now, we see that the boundary of every piece A is an $\epsilon/36$ -locally $\tilde{\eta}$ -Lipschitz curve (if two points of ∂A are closer than $\epsilon/36$, then they are either both on ∂P or both on S_ϵ or both on the cut, or one of them on the cut and another one on S_ϵ or $\partial P \cap \bar{A}$). In all the cases the required inequalities are satisfied according to what we have already demonstrated.

Let us observe now that if some piece A has diameter less than 100 than A is M -elementary for some constant $M = M(\tilde{\eta}, \epsilon)$. Indeed, there exist an $\epsilon/36$ net $X \subset \partial A$ with cardinality $M' = M'(\epsilon)$. Any two elements $x, y \in X$ can be joint by the chain of its elements $x = a_1, a_2, \dots, a_k = y$ such that $d(a_j, a_{j+1}) < \epsilon/36$ and then, from the locally Lipschitz property, by the curve of the length smaller then $M'' = M' \cdot \tilde{\eta}\epsilon$ contained in ∂A . Hence for any two points $s, w \in \partial A$ either $d(s, w) \leq \epsilon/36$ and then there is a curve in ∂A shorter then $\tilde{\eta}d(s, w)$ joining them, or $d(s, w) > \epsilon/36$ and then the curve in ∂A joining s with the closest element $x \in X$, then joining x with $y \in X$ which is the closest to w element of X finally joining y with w has total length not exceeding $(M'' + 2\tilde{\eta})\epsilon < M\epsilon/36 < Md(s, w)$. Hence A is M elementary for $M = 36(M'' + 2\tilde{\eta})$.

It remains to deal with the pieces of diameter greater than 100.

Let A is such a piece. It follows from the construction that $\bar{A} \cap \partial P$ is a broken line of vertical and horizontal segments of which each but the first and the last is adjoint to one angle of 90° and one angle of 270° . W.l.o.g. we may assume that the position of the interior of A is right - up with respect to this broken line, i.e. that 90° angles are in the first quadrant (see Figure 34). Notice that due to the construction and the conditions (v) and (vi) of $P \in VH(\cdot, \frac{1}{2})$ the curve $S_\epsilon \cap \partial A$ consists of vertical and horizontal intervals and the circle arcs which belong to the first

quarter, except possibly for the first and last ones. Let N be a line, which makes 45° with the coordinate axes and pass through the second and forth quadrants. The orthogonal projection of both S_ϵ and ∂P onto N is bi-Lipschitz except for the 200ϵ -neighborhood of the ends of the projection. Let us now choose a point of $\partial P \cap \partial\Omega$ which is projected at the distance of approximately 50 (with the marginal of 10) from the upper end of the projection of A . We make a cut at this point in the direction orthogonal to N . Due to the bi-Lipschitz property of the projection in ϵ -neighborhood of the cut, the cut is regular and the upper components of $A \setminus \overline{py}$ has diameter less than 100, thus being M -elementary, while the other has diameter at least 30 smaller then the diameter of A . Repeating this operation suitable number of times we cut A into M -elementary pieces.

To decompose $\Omega_2 \cup S_{2\epsilon} \cup \Omega_3$ we start by looking at all closed diadic squares with side $\epsilon/16$ which intersect Ω_3 . Denote the union of the squares by $\tilde{\Omega}$. It is clear that $\Omega_3 \subset \tilde{\Omega} \subset \Omega_2 \cup S_{2\epsilon} \cup \Omega_3 \subset \Omega$. Let Ω^* is the complement of the infinite component of $\mathbb{R}^2 \setminus \tilde{\Omega}$. As Ω simply connected, $\Omega^* \subset \Omega$. Also $\partial\Omega^* \subset \partial\tilde{\Omega}$ and so is within $\epsilon/8$ distance from $S_{2\epsilon}$. As each diadic square is M -elementary, it remains only to divide $\Omega_2 \setminus \Omega^*$ into M -elementary pieces.

Observe, that $\partial\Omega^*$ is a simple curve which is $\epsilon/32$ -locally $\sqrt{2}$ -Lipschitz. Let K is a collection of points on $\partial\Omega^*$ which are at the distance between 30 to 40 units from the next one, measured by the length of $\partial\Omega^*$. For a point $z \in K$ let $s \in S_\epsilon$ is such that $d(s, z) = d(S_\epsilon, z)$. Let r is middle of $[s, z]$. Then choose $q \in \partial\Omega^*$ such that $d(q, r) = d(\partial\Omega^*, r)$. Notice, that $\frac{7}{8}\epsilon < d(s, z) \leq \epsilon$ as $d(z, S_{2\epsilon}) < \frac{1}{8}\epsilon$. As well, $d(s, q) \geq d(s, \partial\Omega^*) \geq d(s, \partial S_{2\epsilon}) - d(\partial\Omega^*, S_{2\epsilon}) \geq \frac{7}{8}\epsilon$. By the triangle inequality this implies $d(r, q) > \frac{3}{8}\epsilon$, while $d(r, q) \leq d(r, z) \leq \frac{1}{2}\epsilon$. This, in particular shows that the angle between two intervals of the cut $\kappa = [s, r] \cup [r, q]$ is obtuse.

We claim that the cut $\kappa(z) = [p, r] \cup [r, q]$ is a regular one. Indeed, if $x \in S_\epsilon, y \in \kappa$ and $d(x, y) < \epsilon/36$ then $y \in [s, r]$. As s is the closest to r point of S_ϵ , $d(y, s) \leq d(y, x)$, and by the triangle inequality $d(x, s) \leq 2d(x, y) < \epsilon/12$. Now, by $\epsilon/12$ -locally Lipschitz property of S_ϵ there exists a curve $\gamma_1(x, s) \subset S_\epsilon$, such that $|\gamma_1| \leq \eta d(x, s)$, thus $\gamma(x, y) = \gamma_1 \cup [s, y]$ satisfies $|\gamma| \leq (2\eta + 1)d(x, y)$. The other end of the cut can be treated by a similar argument.

After we remove all the cuts, $\Omega_1 \setminus \Omega^*$ is divided in pieces of diameter less than 100 with $\epsilon/36$ -locally $\tilde{\eta}$ -Lipschitz boundary. As before, those pieces are M -elementary.

This completes the decomposition of P in M -elementary pieces, and thus completes the decomposition of Ω .

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